AD No. 15 837
ASTIA FILE COPY

A THEORETICAL ANALYSIS OF ACOUSTIC WAVE MODES IN LAYERED LIQUIDS

For the
United States Navy Department
Office of Naval Research
Contract N6onr-270, Task Order V
Technical Report No. 9

Department of Electrical Engineering

PRINCETON UNIVERSITY

AD

PRINCETON UNIVERSITY
Department of Electrical Engineering

A THEORETICAL ANALYSIS OF ACOUSTIC WAVE MODES IN LAYERED LIQUIDS

by

Jack L. Stone

Technical Report Number 9
April, 1953

For the
United States Navy Department
Office of Naval Research
Contract N6onr - 270, Task Order V
Reference Number: NR 384 - 201

A THEORETICAL ANALYSIS OF ACOUSTIC WAVE MODES IN LAYERED LIQUIDS

ABSTRACT

The investigation is devoted to a theoretical study of acoustic wave fields arising from a single frequency point source in slightly viscous, layered liquids. The acoustic fields are derived in their modal representation for three particular shallow water configurations: one and two-layered media bounded by perfectly reflecting planes, and the two-layered semi-infinite medium. The resulting mode forms are compared with respect to orthogonality, completeness, finiteness, modal identity and discreteness, physical representations, and "cut-off." Particular attention is given to an analysis and extension of the research of Dr. C. L. Pekeris on the two-layered semi-infinite medium. A derivation of the general power and energy orthogonality conditions, based on studies of electromagnetic wave modes, is also presented.

FOREWORD

The author is particularly indebted to Dr. H. R. Alexander, formerly with the Electrical Engineering Department at Princeton University, now with the Office of Naval Research in Washington, D. C., for his continued guidance, advice and encouragement during the course of this research. He wishes to sincerely thank Professor Walter C. Johnson, chairman of the Electrical Engineering Department at Princeton University, for his support of the investigation.

The contribution of Miss Florence Armstrong, secretary of the Electrical Engineering Department, in the preparation of the manuscript is also gratefully acknowledged.

CONTENTS

CHAPTER 1

INTRODUCTION

One of the most significant advances in the theory of submarine acoustics was the recently published research of Dr. C. L. Pekeris on the propagation of explosive sound in shallow water.[1] In this paper Dr. Pekeris developed a theory of acoustic wave propagation in a layered liquid medium and successfully applied the salient features of the results to some experimental data obtained by Drs. Worzel and Ewing.[2] In his theoretical work, Pekeris represented the fields, arising from a single frequency point source in a horizontal liquid layer underlain by an unbounded bottom of different density and plane-wave velocity, by a sum of acoustic "modes," and a non-vanishing "branch-line" integral. Although the representation was adequate for his practical applications, Dr. Pekeris left unsolved the theoretical problems of modal orthogonality and justification for the existence of the branch line integral. The search for the answers to these questions led to a complete re-examination of the entire problem. The results are embodied in this paper: a new derivation of the field integrals, a new transformation and representation of the final solution, a detailed discussion of the physical interpretation of the mathematical results, and an elaboration of some general properties of acoustic modes in the multi-layered system.

The development begins in Chapter 2 with the derivation of the fundamental acoustic pressure and particle velocity field and wave equations for a vertically stratified, slightly viscous medium. The small viscous effect was carried through only as a first order correction term. A second

---

[1] Pekeris [12]. The bracketed numbers refer to the entries in the reference list.

[2] Worzel [17].

order approximation would include the effect of shear waves in the medium.

In Chapter 3, the wave fields arising from a single frequency point source in three particular media are derived and expressed as a sum of modes. The first two media are one and two layered homogeneous liquids, bounded by perfectly reflecting planes; the third is the Pekeris configuration. The study of the first two problems provides a convenient means for developing relations which can be compared with the Pekeris results. The examination of these problems also suggests the form of the solutions to be expected in multi-layered media. The modes in the three systems are compared with respect to orthogonality, completeness, finiteness, modal identity and discreteness, physical representations, and "cut-off." The derivation of the solution is given at the beginning of each of the sections, and is followed by a descriptive discussion of the more important features of the solution.

Whereas the investigation in Chapter 3 is primarily concerned with the modes as they occur in particular media, Chapter 4 is devoted to the study of some characteristics of acoustic modes that are generally valid. Particular emphasis is placed upon the derivation of the power and energy orthogonality conditions and the physical interpretations of the propagation factors from energy considerations.

Though the problems are restricted to vertically stratified systems in the cylindrical co-ordinate system, the methods can be readily adapted to general wave problems in other co-ordinate systems, as long as the stratification does not lie along the preferred direction of wave propagation. The general source distribution as a function of time and space can be handled by well-known Fourier integral methods combining solutions of

the single frequency point source.

The application of the results given in this paper to the numerical calculation of acoustic fields in shallow water problems is severely limited by the idealized representation of the medium as a system of homogeneous liquid layers. For this reason, it was felt that numerical examples using average oceanic data would be superficial. It is hoped, however, that the discussion will fill the need for a relatively simple explanation of the physical processes occurring in shallow water sound transmission media as represented by these idealized models.

The historical background for this research is contained, for the most part, in Pekeris's article[1] and in the publication of Dr. J. M. Ide, et al.[2] Although his theoretical work has been entirely replaced by Pekeris's contribution, Ide's descriptive material is a good introduction to the problem. The NRC Reports are readable summaries of the general activity in submarine acoustics undertaken by the United States Government during World War II.[3,4]

Several selected references, each having an extensive bibliography, are suggested as complementary reading material: R. B. Adler[5] on the subject of inhomogeneous electromagnetic waveguides, an excellent report which has greatly aided the author; H. W. Marsh, Jr.[6] on anomalous oceanic wave propagation, M.I.T. Radiation Laboratory Series, Volume 13[7] on anomalous radio

---

[1]Pekeris [12], p. 38.
[2]Ide [2].
[3]National Research Council [8].
[4]National Research Council [9].
[5]Adler [1].
[6]Marsh [4].
[7]M.I.T. [5].

wave propagation around the earth, and M. Newlands[1] on the seismology of the two-layered solid medium. A general bibliography on the subject of wave propagation in inhomogeneous media has previously been prepared by the author.[2]

---

[1] Newlands [10].
[2] Stone [15].

CHAPTER II

THE DERIVATION OF THE FUNDAMENTAL EQUATIONS

The orientation of the cylindrical co-ordinate system used in this investigation is shown in Figure 1. Positive z and r are denoted the "depth" and "range" respectively. It is assumed that all fields are independent of the azmuthal co-ordinate θ. The acoustic medium is assumed to be stratified in the sense that its characteristics are functions only of the depth.

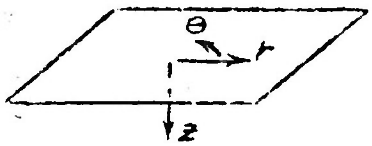

Figure 1.

The symbols are defined as follows:

1. Medium parameters, time and frequency independent:

   $P_o(z) \triangleq$ Equilibrium pressure.

   $\rho(z) \triangleq$ Equilibrium density.

   $c(z) \triangleq$ Proportionality factor for adiabatic wave motion. When constant, this quantity represents the velocity of a plane wave in the unbounded medium.

   $\underline{F}(z) \triangleq$ Vector body force/unit mass.

   $\mu(z) \triangleq$ Viscosity coefficient.

2. Field quantities:

   $\tilde{p}' = P_o(z) + \tilde{p}(r,z,t) \triangleq$ Total instantaneous pressure, where $\tilde{p} \triangleq$ time dependent excess pressure.

   $p' = P_o(z) + p(r,z) \triangleq$ Time independent total pressure, where $p \triangleq$ time independent excess pressure.

   $\tilde{\rho}' = \rho(z) + \rho(z)\,\tilde{s}(r,z,t) \triangleq$ Total instantaneous density, where $\tilde{s} \triangleq$ time dependent condensation.

   $\rho' = \rho(z) + \rho(z)\,s(r,z) \triangleq$ Time independent total density, where $s \triangleq$ time independent condensation.

$\underline{v}$ (r,z,t) $\triangleq$ Instantaneous vector particle velocity.[1]

$\underline{v}$ (r,z) $\triangleq$ Time independent vector particle velocity.

The source-free Navier-Stokes equation for hydrodynamic flow is postulated for regions in which the medium parameters, field variables, and their derivatives are continuous, finite, and single-valued:[2]

$$\tilde{\rho}'\frac{\partial \tilde{\underline{v}}}{\partial t} + \tilde{\rho}'\tilde{\underline{v}}\cdot\nabla\tilde{\underline{v}} = \tilde{\rho}'\underline{E} - \nabla\tilde{p}' + \frac{1}{3}\mu\nabla(\nabla\cdot\tilde{\underline{v}}) + \mu(\nabla\cdot\nabla)\tilde{\underline{v}}. \quad 1.1$$

The following assumptions are reasonable for acoustic problems of interest here:[3]

1. The equilibrium pressure dependence upon depth is negligible. Therefore, from a consideration of the equilibrium state, $\rho\underline{E} = \nabla p_0 \approx 0$.
2. The particle velocity vector is irrotational. This is equivalent to neglecting shear waves, an assumption which is reasonable when the viscous terms in 1.1 are relatively small. Thus,

$$\nabla\times\tilde{\underline{v}} = 0 \quad or \quad \nabla(\nabla\cdot\tilde{\underline{v}}) = \nabla\cdot(\nabla\tilde{\underline{v}}).$$

3. $\tilde{p}, \tilde{s}, \tilde{\underline{v}}$ are small quantities. Products of these terms or their derivatives are negligible to first order.

Equation 1.1 on the basis of these assumptions reduces to:

$$\nabla\tilde{p} = -\rho\frac{\partial\tilde{\underline{v}}}{\partial t} + \frac{4}{3}\mu\nabla(\nabla\cdot\tilde{\underline{v}}), \quad 1.2$$

which is equivalent to Newton's equation of motion in a lossy (viscous) acoustic medium. The equation of mass continuity for a constant-mass system

---

[1]For simplicity, the particle velocity field will often be termed the "velocity-field." and the adiabatic factor c(z) the "velocity parameter."

[2]Page [11], Chap. 6.

[3]Marsh [4], p. 6.

is also postulated:

$$\nabla\cdot\breve{\rho}'\breve{\underline{V}} = -\frac{\partial\breve{\rho}'}{\partial t}. \qquad 1.3$$

By virtue of the previous assumptions, this equation takes the reduced form:

$$\nabla\cdot\rho\breve{\underline{V}} = -\rho\frac{\partial\breve{s}}{\partial t}. \qquad 1.4$$

A relationship between the pressure and the condensation arises from the usual assumption that the wave motion occurs adiabatically, which implies that the excess pressure is proportional to the condensation:

$$\breve{p} = \rho c^2\breve{s}. \qquad 1.5$$

The condensation is eliminated by the combination of equations 1.2, 1.4, and 1.5 whereupon the basic acoustic field equations result:[1]

$$\nabla\breve{p} = -\rho\frac{\partial\breve{\underline{V}}}{\partial t} + \frac{4}{3}\mu\nabla(\nabla\cdot\breve{\underline{V}}), \qquad 1.6a$$

$$\nabla\cdot\rho\breve{\underline{V}} = -\frac{1}{c^2}\frac{\partial\breve{p}}{\partial t}. \qquad 1.6b$$

It is observed that the viscosity effect is found in the force equation but does not enter the equation of continuity.

The generalized "wave" equations are found by performing the divergence operation on equation 1.6a and the gradient operation on equation 1.6b:

$$\nabla^2\breve{p} = \frac{1}{c^2}\frac{\partial^2\breve{p}}{\partial t^2} - \frac{4}{3}\nabla\left(\mu\nabla\left[\frac{1}{\rho c^2}\frac{\partial\breve{p}}{\partial t} + \breve{\underline{V}}\cdot\frac{\nabla\rho}{\rho}\right]\right),$$

$$\nabla(\nabla\cdot\rho\breve{\underline{V}}) = \frac{\rho}{c^2}\frac{\partial^2\breve{\underline{V}}}{\partial t^2} - \frac{4}{3}\frac{\mu}{c^2}\nabla^2\frac{\partial\breve{\underline{V}}}{\partial t} - \frac{\partial\breve{p}}{\partial t}\left(\nabla\frac{1}{c^2}\right),$$

where the substitution:

$$\nabla\cdot\breve{\underline{V}} = -\frac{1}{\rho c^2}\frac{\partial\breve{p}}{\partial t} - \breve{\underline{V}}\cdot\frac{\nabla\rho}{\rho},$$

from equation 1.6b, was introduced. The solution of the wave equations in

[1]Since the particle velocity vector is irrotational, there always exists a velocity potential $\breve{\underline{V}} = \nabla\breve{\phi}$ which can be manipulated as a scalar field. However, for the problems considered here, the use of the velocity potential affords little added convenience.

the general case is clearly a difficult analytical problem because of the complicated interdependence of the field variables. But for the medium consisting of stratified homogeneous liquids the wave equations are simply these:

$$\nabla^2 \tilde{p} - \frac{1}{c^2}\frac{\partial^2 \tilde{p}}{\partial t^2} + \frac{4}{3}\frac{\mu}{\rho c^2}\nabla^2 \frac{\partial \tilde{p}}{\partial t} = 0\,, \qquad 1.7a$$

$$\nabla^2 \tilde{\underline{v}} - \frac{1}{c^2}\frac{\partial^2 \tilde{\underline{v}}}{\partial t^2} + \frac{4}{3}\frac{\mu}{\rho c^2}\nabla^2 \frac{\partial \tilde{\underline{v}}}{\partial t} = 0\,, \qquad 1.7b$$

where the medium parameters are considered constant over a given layer.

For harmonic time variation $\varepsilon^{i\omega t}$, the field and wave equations read:

$$\nabla p = -i\omega\rho \underline{v} - \frac{4}{3}\frac{\omega^2}{c^2}\mu\left(\frac{1}{1+iL}\right)\underline{v} \qquad 1.8a$$

$$\nabla\cdot \underline{v} = -\frac{i\omega}{\rho c^2} p \qquad 1.8b$$

$$\nabla^2 p + \frac{\omega^2}{c^2}\left(\frac{1}{1+iL}\right)p = 0 \qquad 1.8c$$

$$\nabla^2 \underline{v} + \frac{\omega^2}{c^2}\left(\frac{1}{1+iL}\right)\underline{v} = 0\,, \qquad 1.8d$$

where $L \triangleq \frac{4}{3}\frac{\omega\mu}{\rho c^2}$, and equation 1.7b was substituted into equation 1.6a.

In cases of practical interest, the loss factor L is small compared with unity; expansion to first order in L provides the final form of the fundamental equations:

$$\nabla p = -i\omega\rho(1-iL)\underline{v} \qquad 1.9a$$

$$\nabla\cdot \underline{v} = -\frac{i\omega}{\rho c^2} p \qquad 1.9b$$

$$\nabla^2 p + \frac{\omega^2}{c^2}(1-iL)p = 0 \qquad 1.9c$$

$$\nabla^2 \underline{v} + \frac{\omega^2}{c^2}(1-iL)\underline{v} = 0\,. \qquad 1.9d$$

The loss factor thus enters the field equation 1.9a as a "complex" density, and the wave equations 1.9c and 1.9d as a "complex" velocity. The complex velocity parameter is readily interpreted for the case of plane wave propagation in the infinite homogeneous medium. The plane-wave exponential representation $e^{-\gamma x + i\omega t}$ has a propagation factor $\gamma = k - i\alpha$ whose magnitudes can be shown to be:

$$k = \frac{\omega}{c}\left(\frac{[1+L^2]^{\frac{1}{2}}}{2}+\frac{1}{2}\right)^{\frac{1}{2}}, \quad \alpha = \frac{\omega}{c}\left(\frac{[1+L^2]^{\frac{1}{2}}}{2}-\frac{1}{2}\right)^{\frac{1}{2}}.$$

As expected, the loss factor in the complex velocity parameter gives rise to exponential damping and a very small modification in the free space wave number.

The boundary conditions of continuity for the wave fields are specified from purely physical considerations as follows:

Consider two liquid media which are in contact at the equilibrium plane $z = 0$. The medium parameters are specified as shown in Figure 2.

Figure 2. $\quad z=0 \quad \dfrac{\rho_1, c_1, L_1}{\rho_2, c_2, L_2}$

Across any surface in the medium, and, in particular, across the interface $z = 0$:

A. The pressure is everywhere continuous, precluding infinite acceleration of a finite mass. 1.10

B. Except at a source of energy, the normal component of particle velocity is continuous in order that continuous surface contact is insured.

These conditions are sufficient for the determination of the unique solution of equations 1.9. No specification on the tangential component of particle velocity is necessary since shear waves have been neglected.

From equation 1.9a, it is noted that the normal gradient of pressure is discontinuous at a discontinuity of the complex density. This statement is interpreted physically as the fact that a moving interface can support a pressure field which is continuous but may change discontinuously. On the other hand, the fundamental mass continuity equation 1.3 is not defined over the interface at a density discontinuity since the divergence of the discontinuous density-velocity product does not exist, even in the presence of viscous effects. This essentially means that the mass transferred across the interface is not conserved since the moving contact surface involves the same particle displacement of different masses on either side of the boundary. The wave equations, therefore, are not applicable on the interfacial planes between liquids of non-uniform equilibrium density. These peculiarities account for the rather interesting orthogonality conditions to be discussed in Chapter IV.

CHAPTER III

THE SOLUTION OF PARTICULAR PROBLEMS

Having established the fundamental relations among the field variables and the medium parameters, we now concentrate on the determination of the wave fields arising from a single frequency point source in three particular stratified media. The main concern is the derivation and analysis of the fields in the acoustic mode representation. The investigation is aimed toward the following objectives:

1. To extend the analysis of Dr. Pekeris on the mode solutions in the two-layered unbounded medium.

2. To provide a detailed and cohesive study of examples of the three major forms of acoustic modes in vertically stratified liquids.

3. To demonstrate the power of the Fourier-Bessel integral transformation method of solution and the limited application of the well-known orthogonal mode expansion technique.

4. To present, where possiblo, a reasonable explanation of the physical processes involved.

The formal problem is the determination of the pressure and velocity fields due to a point source in a layered liquid medium when the fields satisfy particular conditions at the boundaries and at the source location. It is known that for this type of problem, the unique solution can be found if either the pressure of the normal component of particle velocity is specified over the bounding planes.[1] This paper will be concerned only with the solution of the inhomogeneous form of the pressure wave equation 1.9c and the subsequent determination of the particle velocity field by the use of equation 1.9b.

[1]Stratton [16], p. 485.

One possibly ambiguous aspect of the uniqueness quality of the solution is this: although the solution is uniquely determined by either the pressure or velocity designation on the boundaries, both conditions 1.10 will be applied at non-totally-reflecting surfaces. The difficulty is resolved by noting that these conditions of continuity are concerned with the total resultant field at the boundary, from which we find the fields that exist independently on either side of the boundary. Once the field is found everywhere, the uniqueness theorem states that there is no other possible solution.

It is necessary to add these supplementary conditions on the wave fields to insure uniqueness when the medium extends to infinity:[1]

A. The amplitude of the total field approaches zero at large distances from the source along the medium co-ordinate which extends to infinity. 2.1a

B. Active acoustic energy that is transferred must be diverging from the source when observed at large distances from the source along the medium co-ordinate which extends to infinity. 2.1b

The Single Homogeneous Layer

The single-layered medium bounded by perfectly reflecting planes is shown schematically in Figure 3. The medium represents a shallow-water layer bounded by a zero pressure surface at $z = 0$, and an acoustically "hard"

---

[1] Sommerfeld [14], p. 188.

surface at $z = H$ on which the vertical component of particle velocity is 0. The layer extends to infinity in the radial direction. The density, velocity, and viscosity parameters are constants throughout the medium.

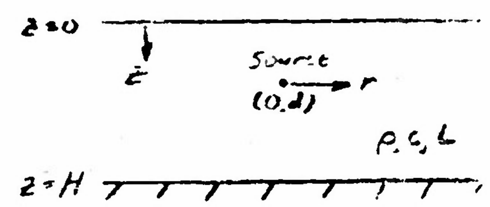

Figure 3.

A point source $Q\,e^{i\omega t}$ which has a total outflow of 1 unit volume/second is located at (0,d). The source as it appears in the inhomogeneous form of the pressure wave equation is represented by the product of Dirac delta functions:[1]

$$Q = -i\omega\rho(1-iL)\,\delta(z-d)\,\delta(r), \tag{2.2a}$$

where:

$$\delta(z-d) = 0,\ z \neq d;\ \int_{-\infty}^{+\infty}\delta(z-d)\,dz = 1 \tag{2.2b}$$

$$\delta(r) = 0,\ r \neq 0;\ \int_{0}^{\infty}\delta(r)\,r\,dr = \frac{1}{2\pi} \tag{2.2c}$$

$$\iiint \delta(z-d)\,\delta(r)\,r\,dr\,d\theta\,dz = 1. \tag{2.2d}$$

The exact statement of the problem is the determination of the solution of the inhomogeneous pressure wave equation:

$$\nabla^2 p + \frac{\omega^2}{c^2}(1-iL)p = -i\omega\rho(1-iL)\,\delta(r)\,\delta(z-d), \tag{2.3}$$

and the determination of the velocity field from the relation:

---

[1]Morse's form, with the substitution of the complex density [6], p. 312.

$$\underline{v} = \frac{\nabla p}{-i\omega\rho(1-iL)}, \qquad 1.9b$$

subject to the boundary conditions:

A. At $z = 0$: $p = 0$ 2.4a

B. At $z = H$: $v_z = \frac{\partial p}{\partial z} = 0$ 2.4b

C. At large r, the total field tends to zero. 2.4c

D. At large r, the field represents divergent radiation, 2.4d

if radiation exists. (The particle velocity in component form is

$\underline{v} = \underline{z}_0 v_z + \underline{r}_0 v_r$, where $\underline{z}_0$ and $\underline{r}_0$ are unit vectors along the positive direction of the co-ordinate axes.)

We shall first solve this problem by the method of orthogonal modes which can always be used in the homogeneous layer configuration when the homogeneous conditions are satisfied over the bounding planes.[1]

$$A(\omega)p + B(\omega)v_z = 0 . \qquad 2.5$$

This equation represents the familiar impedance boundary condition which will be briefly discussed later. The method is outlined as follows:

1. An infinite set of discreet solutions of the source-free wave equation is found, all members of the set individually satisfying the boundary conditions. Each of these solutions is a "mode."

2. These modes are shown to comprise an orthogonal set of functions whose properties provide a representation of the source term by a linear combination of the modes.

3. The pressure field is then found to be another linear combination of these modes, whose amplitudes are evaluated in terms of the source expansion

[1] Sommerfeld [14], p. 168.

coefficients.

In cylindrical co-ordinates, the homogeneous form of equation 2.3 is:

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{\partial^2 p}{\partial z^2} + \frac{\omega^2}{c^2}(1 - iL)p = 0 . \qquad 2.6$$

If it is assumed that the pressure field can be expressed in the separated form:

$$p(r,z) = R(r)F(z) , \qquad 2.7$$

then equation 2.6 is directly reduced to two ordinary differential equations:

$$\frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \gamma^2 R = 0 \qquad 2.8a$$

$$\frac{d^2 F}{dz^2} + \left(\frac{\omega^2}{c^2}[1 - iL] - \gamma^2\right)F = 0 , \qquad 2.8b$$

where γ^2 is the separation constant. For later convenience, let:

$\gamma \triangleq k + i\alpha$, k and α real, $\gamma \triangleq$ Propagation factor.

$\beta^2 \triangleq \left(\frac{\omega^2}{c^2}[1 - iL] - \gamma^2\right)$, $\beta \triangleq$ Distribution factor.

The z co-ordinate boundary conditions are now applied to 2.8b. The result is an infinite number of discrete solutions of 2.6, defined by the relations:

$$P_n = R_n(r)F_n(z) , \quad n = 1, 2, --, \infty \qquad 2.9a$$

$$F_n = \sin \beta_n z \qquad 2.9b$$

$$\beta_n = (n - \tfrac{1}{2})\frac{\pi}{H} \qquad 2.9c$$

$$\gamma_n^2 = \frac{\omega^2}{c^2}(1 - iL) - \beta_n^2 . \qquad 2.9d$$

Now let us assume that the z dependent delta function can be expanded

into an infinite series of the F-functions:

$$\delta(z-d) = \sum_n A_n F_n(z). \qquad 2.10$$

The constants A_n can be evaluated by application of the orthogonal properties of the F-functions:

$$\int_{-\infty}^{+\infty} F_m F_n \, dz = \int_0^H F_m F_n \, dz = \begin{matrix} 0, & m \neq n \\ \frac{H}{2}, & m = n \end{matrix} , \qquad 2.11$$

where the fields in the range $z<0$, $z>H$ are everywhere 0. An integration of equation 2.10 over the range of z determines the constants:

$$A_n = \frac{2}{H}\int_0^H \delta(z-d) F_n(z) dz = \frac{2}{H} F_n(d). \qquad 2.12$$

This series representation of the delta function, however, does not converge for any value of z. The justification for its use lies in the fact that the final representation of the field as a series does converge for all r and z except at the source singularity r = 0.[1]

The solution of the inhomogeneous wave equation 2.6 is now assumed to be a linear combination of modes:

$$p(r,z) = \sum_n R_n(r) F_n(z), \qquad 2.13$$

in which each of the functions R_n contains an unknown constant factor which will be evaluated in terms of the source expansion coefficients. Introducing this expansion and the preceding relations into the wave equation 2.6, we can produce the following equations:

$$\sum_n F_n(z)\left(\frac{d^2R_n}{dr^2} + \frac{1}{r}\frac{dR_n}{dr} + \gamma_n^2 R_n\right) = -i\omega\rho(1-iL)\delta(r)\sum_n F_n(z)F_n(d). \qquad 2.14$$

---

[1]A similar situation exists in the vibrating string problem discussed by Morse [6], p. 93.

The application of the orthogonality conditions 2.11 provides the equations for the R-functions:

$$\frac{d^2R_n}{dr^2} + \frac{1}{r}\frac{dR_n}{dr} + \gamma_n^2 R_n = \frac{2}{H}\left(-i\omega\rho[1-iL]\right)\delta(r)F_n(d).$$

Let:
$$R_n = \frac{2}{H}\left(-i\omega\rho[1-iL]\right)F_n(d)H_n(r).$$

Then:
$$\frac{d^2H_n}{dr^2} + \frac{1}{r}\frac{dH_n}{dr} + \gamma_n^2 H_n = \delta(r). \qquad 2.15$$

The H_n functions are the Green's functions for the cylindrical system which represent diverging radiation and vanishing amplitude at large ranges, and satisfy the singularity requirement at r = 0.[1] Corresponding to the time factor $e^{i\omega t}$, the appropriate solutions of 2.15 are the zero order Hankel Functions of the second kind with a normalizing factor of πi. That this Hankel function represents diverging radiation is seen from its asymptotic form at large ranges:

$$H_0^{(2)}(\gamma_n r)e^{i\omega t} \xrightarrow[r\to\infty]{} \sqrt{\frac{2}{\pi}}\frac{e^{-i\gamma_n r}}{\sqrt{\gamma_n r}}e^{i\omega t}. \qquad 2.16$$

The pressure and velocity fields are now completely determined:

$$p = \frac{2\pi i}{H}\left(-i\omega\rho[1-iL]\right)\sum_{n=1}^{\infty}\sin\beta_n d\,\sin\beta_n z\,H_0^{(2)}(\gamma_n r) \qquad 2.17a$$

$$\underline{v} = \frac{2\pi i}{H}\sum_{n=1}^{\infty}\sin\beta_n d\,\nabla\left(\sin\beta_n z\,H_0^{(2)}(\gamma_n r)\right). \qquad 2.17b$$

A discussion of the solution will be undertaken later.

We shall now solve the preceding problem by the Fourier-Bessel integral method which alone will be applicable in the later problems. This method is considerably more powerful than the method of orthogonal modes since modal orthogonality, among other qualities of the modal set, is not a

---

[1] Morse [7].

prerequisite for a solution but rather a consequence. The technique in outline form is as follows:

1. The point source term is expressed as a Fourier-Bessel transform which physically represents a linear combination of a continuum of plane sources.

2. The pressure field due to the presence of one plane source is found.

3. The field of the infinite continuum of plane sources acting simultaneously are linearly combined to give the field of the point source.

Any reasonable function of the range co-ordinate can be expressed in an integral form:

$$f(r) = \int_0^\infty g(k) J_0(kr)\, k\, dk \qquad 2.18$$

when the integral exists.[1] $J_0(kr)$ is the zero-order Bessel function of the first kind, k is purely real, and g(k), defined as the Fourier-Bessel transform, is found from the inverse transformation:

$$g(k) = \int_0^\infty f(r) J_0(kr)\, r\, dr \qquad 2.19$$

Let us express the radial delta function as a Fourier-Bessel integral. Its transform is:

$$g(k) = \int_0^\infty \delta(r) J_0(kr)\, r\, dr = 1 \qquad 2.20$$

and the final expression for the Fourier-Bessel integral is:

$$\delta(r) = \int_0^\infty J_0(kr)\, k\, dk \qquad 2.21$$

It is easy to show from an integration by parts that the above integral does,

[1] Sommerfeld [14], p. 240.

in fact, diverge for all r. This representation, however, will be used by analogy with the previous case of the non-converging series representation of the z delta function, where the justification is, again, that the final expression is everywhere convergent except at the source axis, $r = 0$.

The source term of equation 2.3 in its integral form is:

$$Q = \int_0^\infty -i\omega\rho(1-iL)\delta(z-d)J_0(kr)k\,dk. \qquad 2.22$$

We can give a direct physical interpretation to the above expression. The integrand represents a plane acoustic source at $z = d$, with a radial amplitude distribution of magnitude $kJ_0(kr)$. Figure 4 is a schematic representation of the plane source distribution.

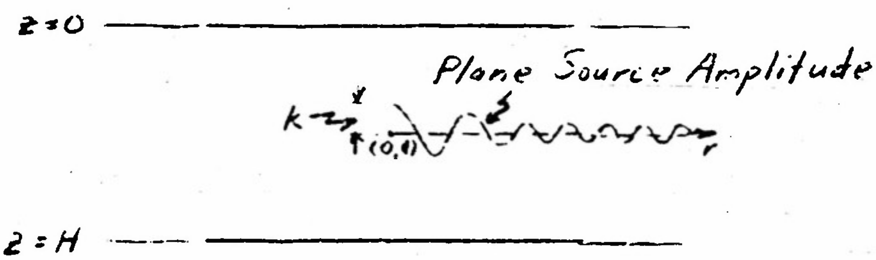

Figure 4.

Let us now assume that a single plane source exists in the medium at the plane $z = d$ with a radial amplitude distribution corresponding to the number k. We shall first find the fields due to this source by solving the wave equation:

$$\nabla^2 p + \frac{\omega^2}{c^2}(1-iL)p = -i\omega\rho(1-iL)\delta(z-d)kJ_0(kr), \qquad 2.23$$

subject to the same boundary conditions as before, equations 2.4.[1] The pressure field is assumed to be composed of two terms: $p_p + p_c$. p_p is the

[1]This method was suggested by the derivation given by Adler for the case of electromagnetic radiation from a dipole in a cylindrical dielectric rod which exists in free space [1], App. 4.

"particular" solution of the wave equation; it completely represents the presence of the source, and only the source, and is chosen as though the medium were unbounded. ϕ_c is the "complementary" solution of the source free wave equation and is so combined with ϕ_p that the <u>total</u> field satisfies the boundary conditions at the reflecting planes. It is explicitly stated that neither of these terms satisfies the boundary conditions independently.

The field transition conditions across the source plane must first be established. Let the fields above and below the plane z = d be denoted by the subscripts 1 and 2 respectively. Since the pressure is everywhere continuous,

$$\text{At } z = d: \ \phi_{p1} = \phi_{p2} \,. \tag{2.24}$$

However, the particle velocity vector is not continuous because the fluid is moving in opposite directions above and below the source. The magnitude of the velocity discontinuity is found from the fact that the vertical component of particle velocity at the source surface is equal to the total volume outflow/second in the z direction. Since the delta function represents a volume outflow/unit volume/second, the total outflow in the z direction is simply:

$$\int_0^{H} \delta(z-d)\, k J_0(kr)\, dz = k J_0(kr) \,. \tag{2.25}$$

The normal component of particle velocity at the source surface, therefore, has the magnitude:

$$|v_z| = k J_0(kr) \,.$$

Consequently, the velocity discontinuity at the source plane is:

$$\text{At } z = d: \ |v_{z2}| - |v_{z1}| = -2k J_0(kr) \,.^{1} \tag{2.26}$$

---

[1] The minus sign is chosen for correspondence with the sign of the downward traveling pressure wave in equation 2.27b.

The particular solution ϕ_p can now be found by assuming appropriate solutions of the wave equation 2.24 in the regions above and below the source, and matching the fields at the plane z = d. Since the boundary conditions will be applied to the total field, we can choose a solution which represents upward traveling waves above the source and downward traveling waves below the source. The acceptable choice is:

$$\phi_{p1} = A e^{i\beta(z-d)} J_0(kr) \quad , 0 \leq z \leq d \tag{2.27a}$$

$$\phi_{p2} = B e^{-i\beta(z-d)} J_0(kr), \quad d \leq z \leq H, \tag{2.27b}$$

provided that:

$$\beta^2 = \frac{\omega^2}{c^2}(1-iL) - k^2. \tag{2.28}$$

The constants A and B are evaluated by matching the above equations over the plane z = d by the application of equations 2.24 and 2.26. The particular solution is found to be:

$$\begin{aligned} \phi_p &= -i\omega\rho(1-iL)\frac{e^{i\beta(z-d)}}{i\beta} k J_0(kr), \quad 0 \leq z \leq d \\ &= -i\omega\rho(1-iL)\frac{e^{-i\beta(z-d)}}{i\beta} k J_0(kr), \quad d \leq z \leq H. \end{aligned} \tag{2.29}$$

The complementary solution is simply chosen as

$$\phi_c = M e^{i\beta z} J_0(kr) + N e^{-i\beta z} J_0(kr), \tag{2.30}$$

which represents, respectively, upward and downward traveling waves in the absence of a source.

The constants are found directly from the application of the z-co-ordinate boundary conditions 2.4a and b to the sum of ϕ_p and ϕ_c:

$$M = \frac{D k e^{-i\beta H} \sin\beta d}{\beta \cos\beta H} \; ; \; N = \frac{-D k \cos\beta(H-d)}{i\beta\cos\beta H}, \tag{2.31a}$$

where

$$D \triangleq -i\omega\rho(1-iL) \tag{2.31b}$$

The conditions 2.4c and d will be applied only to the final solution since they are compatible only with source distributions located entirely within a finite radius of the origin.[1]

The fields of the original point source are found by integrating equation 2.23 over the complete range of k; that is to say, we sum the pressure fields due to the contributions of an infinite continuum of plane sources operating simultaneously over the plane z = d:

$$\nabla^2\int_0^\infty p\,dk + \frac{\omega^2}{c^2}(1-iL)\int_0^\infty p\,dk = -i\omega\rho(1-iL)\,\delta(z-d)\,\delta(r). \qquad 2.32$$

The field of the point source in the single-layered medium is, then:

$$p = -i\omega\rho(1-iL)\left(\int_0^\infty \frac{e^{\pm i\beta(z-d)}}{i\beta}kJ_0(kr)\,dk + \int_0^\infty \frac{e^{-i\beta H}\sin\beta d\, e^{i\beta z}}{\beta\cos\beta H}kJ_0(kr)\,dk \right.$$
$$\left. - \int_0^\infty \frac{\cos\beta(H-d)e^{-i\beta z}}{i\beta\cos\beta H}kJ_0(kr)\,dk\right), \quad \begin{matrix} +: 0 \leq z \leq d \\ -: d \leq z \leq H \end{matrix}. \qquad 2.33$$

The first integral must obviously represent the source completely and should be expected to reduce to the field of a point source in the unbounded medium. That this actually occurs is shown by an extension of the Sommerfeld transformation to the lossy case:[2]

$$\int_0^\infty \frac{e^{\pm i\beta(z-d)}}{i\beta}kJ_0(kr)\,dk = \frac{e^{-i\left(\frac{\omega^2}{c^2}[1-iL][r^2+(z-d)^2]\right)^{1/2}}}{[r^2+(z-d)^2]^{1/2}}. \qquad 2.34$$

For ease in manipulation, the expressions for the total field can be contracted into the following forms:

---

[1] Stratton [16], p. 485.

[2] Sommerfeld [14], p. 240

$$p = \int_0^\infty \frac{2DJ_0(kr)kdk}{\beta\cos\beta H}\sin\beta z(\cos\beta H\cos\beta d + \sin\beta H\sin\beta d), \quad 0 \le z \le d$$

$$p = \int_0^\infty \frac{2DJ_0(kr)kdk}{\beta\cos\beta H}\sin\beta d(\cos\beta H\cos\beta z + \sin\beta H\sin\beta z), \quad d \le z \le H \qquad 2.35$$

or, in the abbreviated fashion:

$$p = \int_0^\infty 2G_{1,2}(\beta)J_0(kr)kdk. \qquad 2.36$$

The remainder of the derivation is devoted to the exact interpretation and evaluation of the field integrals 2.35. Considered as functions of the real variable k, the integrands are not analytic everywhere due to the presence of branch points and simple poles. We shall therefore extend the integrand, by means of the principle of analytic continuation, into the complex plane where the integral can be properly defined by an appropriate choice of a path in the complex k plane. To this end, the complex variable $\gamma = k + i\alpha$, k and α real, is substituted for the real variable k.

The major objective is to transform the existing integrals into line integrals along closed paths about the poles of the integrand in such a manner that only integrals along the pole contours contribute non-vanishing terms. Particular attention is given to circular contours with large radii on which the integrands provide vanishing contributions as the radii approach infinity. In their present formulation, the integrals will nowhere vanish for large γ because the Bessel function grows in magnitude for large values of a complex argument. The transformation

$$2J_0(\gamma r) = H_0^{(1)}(\gamma r) + H_0^{(2)}(\gamma r) \qquad 2.37$$

is introduced to insure that the integrands will vanish for large γ. From their asymptotic forms,

$$H_0^{(1)}(\gamma r) \sim e^{i\gamma r}, \quad H_0^{(2)}(\gamma r) \sim e^{-i\gamma r}, \qquad 2.38$$

it is apparent that the Hankel function of the first kind will vanish on an infinite contour in the upper half γ-plane, and the Hankel function of the second kind will vanish on an infinite contour in the lower half γ-plane.

We shall next clear the matter of the double-valuedness of the integrands of 2.35 due to the double choice of the sign of β, as seen from its definition in equation 2.28. Each choice of sign defines, essentially, one branch of a two-sheeted Reimann surface.[1] The sign of the root, and hence the choice of sheet on which the integration path is to lie, is chosen so that the integrals 2.35 will converge for large k by insuring that the integrands rapidly approach zero as k approaches infinity.[2] Except near the branch point $\beta=0$, it is observed that β is essentially imaginary for large k. The positive sign of the root is thus indicated with β defined as follows:

$$\begin{aligned} &\text{For } 0<k<\frac{\omega}{c}: \beta = \left(\frac{\omega^2}{c^2}[1-iL] - k^2\right)^{1/2}, \arg\beta \lesssim 0 \\ &\quad k>\frac{\omega}{c}: \beta = -i\left(k^2-\frac{\omega^2}{c^2}[1-iL]\right)^{1/2}, \arg\beta \simeq -\frac{\pi}{2} \end{aligned} \qquad 2.39$$

Noting the restrictions on the ranges of z, one can easily show that the integrands of 2.35 will assuredly vanish for large k.

It is now necessary to insert a "branch" cut into the complex plane in order that the analyticity of the integrand is preserved. The cut starts at the branch point and extends to infinity; as long as the phase specifications on β in equations 2.39 are adhered to, the cut can be oriented in any direction. A tentative choice is a cut parallel to the imaginary axis as shown in Figure 5, where the small imaginary part of β has been neglected in the drawing.[3] Approximate phases of β are given at four points near the cut.

---

[1] Morse [7].

[2] Sommerfeld [14], p. 251.

[3] This particular cut is suggested by Pekeris's method [12], Part 2.

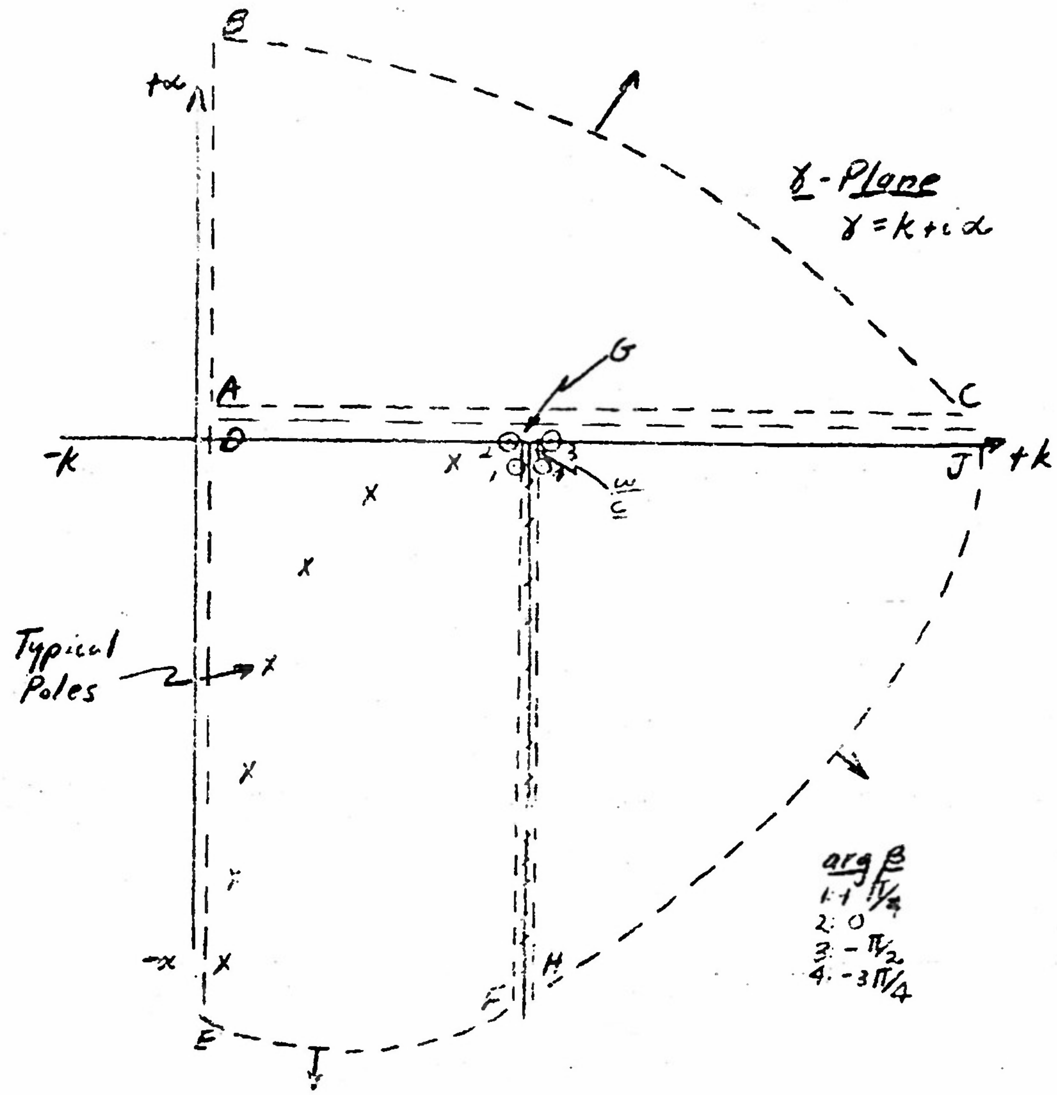

Figure 5.

Regardless of the branch cut orientation, the argument of β changes 180° across the cut. This is easiest seen by expressing β in the form:

$$\beta = \left(\frac{\omega^2}{\underline{c}^2} - \gamma^2\right)^{1/2} = \left(\frac{\omega}{\underline{c}} - \gamma\right)^{1/2}\left(\frac{\omega}{\underline{c}} + \gamma\right)^{1/2}, \quad \underline{c} = \frac{c}{(1-iL)^{1/2}}.$$

Defining the complex vector $he^{i\phi}$, h and ϕ real, as $\frac{\omega}{\underline{c}} - \gamma$, we observe that:

$$\beta = h^{1/2} e^{\frac{i\phi}{2}} \left(\frac{\omega}{\underline{c}} + \gamma\right)^{1/2}.$$

It is clear that ϕ changes by 2π radians for choices of γ on either side and adjacent to the branch cut.

The next operation is the demarcation of the poles of the integrands of 2.35 which lie at the values of γ for which:

$$\cos\beta_n H = 0, \quad n = 1, 2, \cdots, \infty.$$

The poles are these:

$$\beta_n = (n - \tfrac{1}{2})\frac{\pi}{H},$$

$$\gamma_n^2 = k_n^2 - \alpha_n^2 + 2ik_n\alpha_n = \frac{\omega^2}{c^2}(1 - iL) - \beta_n^2. \qquad 2.40$$

All of the poles in the right half of the plane lie in the fourth quadrant in the region $k < \omega/c$. From the imaginary part of the above equation it is seen that the poles lie on a hyperbola which approaches infinity along the negative imaginary axis. In the lossless case, there exist a finite number of poles on the real axis, and an infinite number on the negative imaginary axis.

The field integrals will now be defined in the complex γ-plane of Figure 5. Introducing the Hankel function transformation, 2.37, the integrals in 2.36 take the form:

$$p = \int_0^\infty G(\beta) H_0^{(1)}(kr) k\,dk + \int_0^\infty G(\beta) H_0^{(2)}(kr) k\,dk.$$

Cauchy's Theorem provides the following representation of the above equations:

$$\int_0^\infty G(\beta) H_0^{(1)}(kr) k\,dk = \int_{AB+BC} G(\beta) H_0^{(1)}(\gamma r)\gamma\,d\gamma,$$

$$\int_0^\infty G(\beta) H_0^{(2)}(kr) k\,dk = \int_{DE+EF+FG+GH+HJ+\text{Poles}} G(\beta) H_0^{(2)}(\gamma r)\gamma\,d\gamma,$$

in which the direction along the path is given by the order of the subscript letters, and the last integral represents the sum of the integrals on contours encircling all the poles in the fourth quadrant, taken in the clockwise direction. The paths CB, FE, and JH are chosen to be segments of a circle

whose center is at the origin. As the radius of the circle tends to infinity, the integrands along the paths BC and HJ become vanishingly small for all z. and all $r \neq 0$. Along EF however, the integrand will vanish only when the range is at least of order of the depth and of the wavelength since the imaginary part of β is positive.[1] This restriction is rather awkward; a representation which is everywhere valid is more desirable.[3] This aim can be achieved by choosing the branch-cut orientation so that the imaginary part of β is negative over the entire right half γ-plane.[2] The remainder of this paragraph is restricted to the lossless case. It is easily shown from equations 2.[illegible]9 that the branch cut should lie along the path $Im(\beta) = 0$; more precisely, the branch cut is along the real axis from 0 to ω/c. and along the negative imaginary axis from 0 to ∞. as shown in Figure 6.

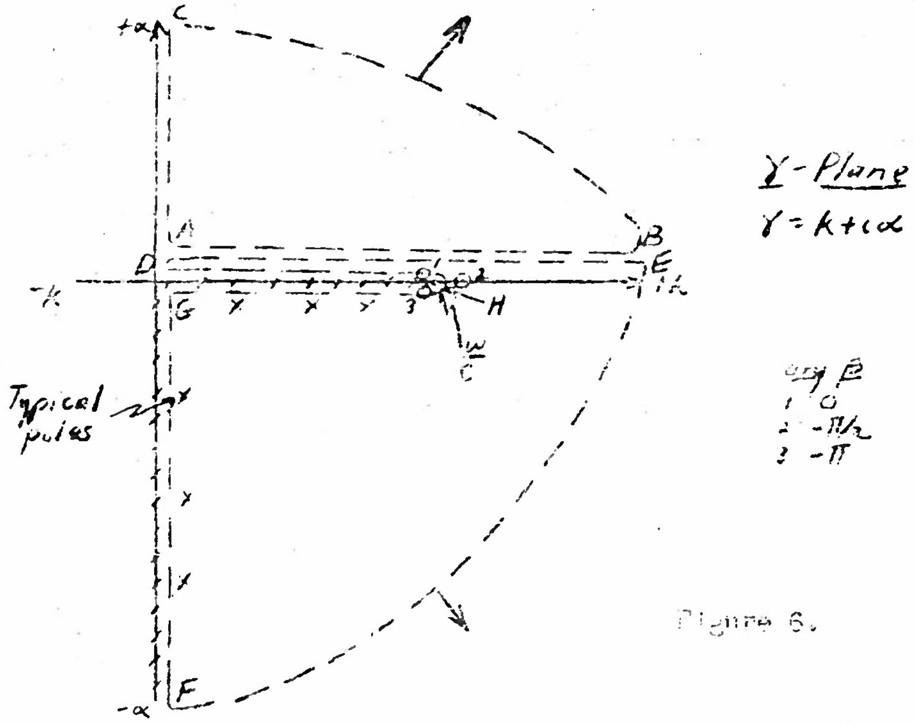

Figure 6.

---

[1] See equation 2.3[illegible].

[2] Suggested by the work of Lapwood [illegible].

[3] From the viewpoint of practical calculation of near-field phenomena. The question of the validity of the Fourier-Bessel point source representation (2.21) also arises when the solution is not everywhere applicable.

The phases of β are also indicated on this figure. The poles are considered to lie an infinitesmal distance away from the branch cuts by the assumption that an infinitesimal loss is present, however slight. Cauchy's Theorem then provides this representation of the integrals of equations 2.35, referring the contour designations to Figure 6:

$$\int_0^\infty G(\beta) H_0^{(1)}(kr) k\,dk = \int_{AC+CB} G(\beta) H_0^{(1)}(\gamma r)\gamma\,d\gamma$$

$$\int_0^\infty G(\beta) H_0^{(2)}(kr) k\,dk = \int_{DH+HG+GF+FE+Poles} G(\beta) H_0^{(2)}(\gamma r)\gamma\,d\gamma\,.$$

The paths CB and FE are again chosen to be segments of a circle whose center is at the origin, and the integrands along these paths tend to zero as the radius approached infinity for all z and all $r \neq 0$. The sum of the integrals along AC and GF are shown to vanish by Pekeris.[1] The integrals along the branch line:

$$\int_{DH+HG} G(\beta) H_0^{(2)}(kr) k\,dk = \int_{DH} [G(\beta) - G(-\beta)]\, H_0^{(2)}(kr) k\,dk = 0,$$

since β simply changes sign on either side of the cut and the integrand is even with respect to β. The only remaining contribution to the field comes from the line integrals around the poles, which, of course, is the sum of the residues of the integral; thus

$$p = 2\pi i \, Res\left[G(\beta) H_0^{(2)}(\gamma r)\gamma\right].$$

The residue at the nth pole γ_n is given by the application of a well-known theorem:

$$Res._n = \frac{F(\beta_n) H_0^{(2)}(\gamma_n r)\gamma_n}{\left.\frac{\partial}{\partial\gamma}\cos\beta H\right|_{\gamma=\gamma_n}}, \qquad 2.41$$

[1] Pekeris [12], Part 2.

where: $$G(\beta) = \frac{E(\beta)}{\cos\beta H} .$$

The residues can be evaluated easily, and the result is the same for fields both above and below the source:

$$p = \frac{2\pi i}{H}\left(-i\omega\rho[1-iL]\right)\sum_{n=1}^{\infty} \sin\beta_n d \sin\beta_n z\, H_0^{(2)}(\gamma_n r). \qquad 2.42$$

This expression is precisely that derived by the method of orthogonal modes, equation 2.17a. The particle velocity field is, of course, also the same. This correspondence attests to the validity of the Fourier-Bessel method.

The effect of a small loss does not change the derivation significantly. A branch cut can always be found such that the imaginary part of β is negative over the right half γ-plane; it takes the form of a segment of a hyperbola starting at the branchpoint, and extending to infinity along the negative imaginary axis. The line integrals along all the contours except the pole contours will vanish for exactly the same reasons as outlined above. The residues have the same form as equation 2.42, except now the poles are all complex.

The solution will now be discussed in detail.

1. A comparison of the two methods of solution indicates that those characteristics of the modal field representation which are necessary to successfully apply the method of orthogonal modes are actually consequences of the solution when derived by the Fourier-Bessel integral method. These characteristics are completeness, orthogonality, and satisfaction of boundary conditions.[1]

A. Completeness of a modal set essentially provides the guarantee

[1] Suggested practical references for these concepts are the investigations of Sommerfeld [14] and Morse [8].

that an arbitrary, continuous acoustic field in the medium can be exactly represented by a linear combination of the modes. The set of modes was assumed to be complete in the orthogonal mode method, whereas completeness of the modal set was insured from the integral transformation method in view of the disappearance of all contour integrals but the residue components. We shall later find that the set of discrete modes is not complete when the medium is unbounded in the z direction.

B. Modal orthogonality is, of course, the major prerequisite for the application of the orthogonal mode technique. This property, however, is clearly a by-product of the solution by the integral transformation.

C. A further requirement for the use of the orthogonal mode method is the necessity that each of the modes satisfies the boundary conditions independently. In the Fourier-Bessel method, the initial form of the solution contained two terms which did not independantly satisfy the boundary conditions but eventually were transformed into the modal set. It is thus reassuring but not requisite that the integral transformation should result in an infinite set of discrete solutions of the wave equation. This point is quite significant for solutions in medium configurations which are unbounded in the z direction.

D. As long as the medium is homogeneous, and subject only to the homogeneous boundary conditions, we can say that orthogonality and completeness imply one another. However, in the multi-layer, or inhomogeneous case, we must examine the final results of the integral transformation method in the individual case before a judgment is made. In the next two problems we shall find, respectively, examples of completeness without orthogonality, and orthogonality without completeness.

2. It is noteworthy that the final solution determined by the Fourier-Bessel

integral transformation is essentially independent of the choice of branch cut since the pole contours alone contributed to the solution. When the medium is unbounded in the z directions, we shall pay this severe penalty for extending the integral into the complex plane: the field representation will depend upon the choice of branch cut, resulting in an infinite number of available representations.

3. Several physical interpretations of the individual modes are possible when the Hankel function is approximated by its asymptotic form:

$$\tilde{p}_n \sim \sin\beta_n z \frac{e^{-\alpha_n r - i k_n r} e^{i\omega t}}{(k_n - i\alpha_n)^{1/2} r^{1/2}} . \qquad 2.43$$

It is clear that the mode represents an inhomogeneous (z dependent) cylindrical wave, which is diverging from the source. This wave propagates with the phase velocity $\frac{\omega}{k_n}$ and damps exponentially at the rate of α_n nepers/unit range, in addition to the geometrical spreading indicated by the factor $r^{-1/2}$. In the lossless medium, there are a finite number of modes for which $\alpha_n = 0$. A term of this nature represents an undamped cylindrical wave which, as shown from considerations in Chapter IV, radiates energy without loss. A term for which $k_n = 0$ represents an oscillatory, damped field which is easily shown to carry no real energy.

At large ranges where the factor $r^{-1/2}$ can be neglected when compared to the exponential factor, the undamped or freely propagating mode can be interpreted as the resultant of two constructively interfering traveling "plane" waves. For a given mode, the plane waves are traveling at discrete angles θ_n with respect to the vertical given by the well-known relation:

$$\sin\theta_n = \frac{k_n c}{\omega} . \qquad 2.44$$

We thus find that the plane sources with propagation factors k_n operate into a "resonant" condition in which energy can be transmitted without loss. The integrated effect of all other plane sources is destructive interference with no resultant radiated energy.

4. It was found that second quadrant poles, representing converging waves did not appear in the solution because the contours did not leave the right half γ-plane. There were no first or third quadrant poles of the field integrands. These poles, which correspond to waves whose amplitudes increase exponentially in the radial direction are excluded as a direct consequence of the fundamental equations and not as a result of a separately imposed boundary condition. In Chapter IV, this statement is proved to be applicable in the multi-layered system.

5. It is recalled that, even in the generally lossy case, the distribution parameter β_n is real and a function only of the layer depth H. The modal dependence upon the vertical co-ordinate is consequently a purely sinusoidal function which is fixed when the layer depth is specified. It is therefore possible to use the β_n, and the resulting vertical field distribution, as a label identifying the nth mode. It is particularly important to note that the modes are clearly identified when considered a function of source frequency.

6. Let us again study the lossless case. From equation 2.40, it is observed that all of the poles on the real axis fall in the range $0 < k_n < \frac{\omega}{c}$, for all frequencies. It is also clear that k_n decreases from $\frac{\omega}{c}$ at the high frequencies to 0 at a certain finite frequency, which is defined as the "cut-off" frequency of the nth mode. Below the cut-off frequency, the pole on the real axis disappears, and a new pole, corresponding to the <u>same</u> β_n, appears on the imaginary axis; the mode thus changes from free propagation

to a purely damped field.

By comparison with equation 2.44, "cut-off" can be described as follows: At a finite frequency, there exist a finite number of sets of constructively interfering plane traveling waves, propagating at angles θ_n. For the nth set, the waves travel at near grazing incidence ($\theta_n \approx 90^\circ$) at the high frequencies, to normal incidence ($\theta_n \approx 0^\circ$) at the cut-off frequency. Below cut-off, the constructive interference is no longer possible (θ_n becomes imaginary). The meaning of the cut-off frequency is especially precise in this problem because the mode is clearly identified by the β_n tag. The situation is radically different in the multi-layered systems and will be discussed in detail later.

7. The wave propagation exhibits phase velocity dispersion through the dependence of K_n on frequency (equation 2.40). Since the phase velocity $V_n = \frac{\omega}{K_n}$, it is clear from equation 2.40 and the discussion above that the phase velocity is ∞ at cut-off, and decreases monotonically towards C at the high frequencies. The velocity of energy propagation or group velocity, always different from the wave velocity in a dispersive system, is derived in Chapter IV.

The nature of the propagation factor K_n also provides us with an explanation for the appearance of a branch point. The possibility of K_n having the value at the branch point, which is just the propagation factor of a plane wave in the unbounded medium, implies the possible existence of a wave which is not influenced by the boundaries.

8. The familiar representation of the acoustic fields by image source contributions is derivable from an expansion of the denominators of the field integrals, equations 2.35. Dr. Pekeris has shown that the ray series and

mode series are related by a Poisson summation.[1] "Image-type" representations of fields in the multi-layered system are possible but will not be discussed in this paper, since Dr. Pekeris has examined them in detail.[2]

9. The solution for the modes in the system with the general homogeneous or "impedance" boundary condition has the same form as the solution given in equation 2.42. This condition represents a fixed relation between the pressure and particle velocity fields over the bounding plane, and consequently, can exist only when the wave phase fronts are approximately plane or where "ray" theory applies. Dr. Pekeris has derived an expression for the limit of applicability of the impedance boundary approximation in the layered liquid medium.[3]

The Two-Layered Medium Bounded by Perfectly Reflecting Planes

The configuration for the bounded two-layer medium is shown in Figure 7. The upper layer represents a shallow water layer bounded by a zero pressure surface at $z = 0$. The lower layer represents an idealization of the ocean bottom in which it is assumed to be a liquid with appropriate acoustic constants. The liquids are in continuous contact at the interface $z = H$. The lower layer is bounded by an acoustically "hard" surface at $z = H + h$. A point source of energy is located at the point $(0,d)$.

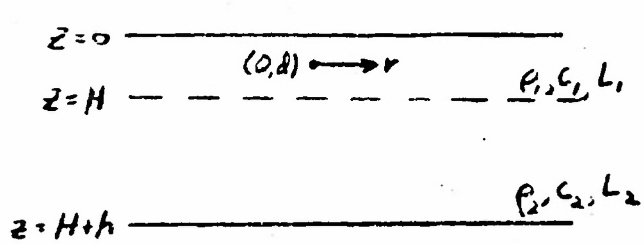

Figure 7.

[1] Pekeris [13].

[2,3] Pekeris [12], Part 2.

The constants of the lower layer are assumed to be greater than the corresponding constants of the upper layer. The subscripts 1 and 2 refer to quantities in the upper and lower layers respectively.

The exact statement of the problem is the search for the solution of the pressure wave equations in each of the layers:

$$\nabla^2 p_1 + \frac{\omega^2}{c_1^2}(1-iL_1)p_1 = -i\omega\rho_1(1-iL_1)\delta(r)\delta(z-d), \quad 0 \leqq z \leqq H \tag{2.45a}$$

$$\nabla^2 p_2 + \frac{\omega^2}{c_2^2}(1-iL_2)p_2 = 0, \quad H \leqq z \leqq H+h, \tag{2.45b}$$

and the determination of the particle velocity fields:

$$\mathbf{v}_1 = \frac{\nabla p_1}{-i\omega\rho_1(1-iL_1)} \tag{2.46a}$$

$$\mathbf{v}_2 = \frac{\nabla p_2}{-i\omega\rho_2(1-iL_2)}, \tag{2.46b}$$

subject to the boundary conditions:

A. At $z = 0$: $p_1 = 0$. 2.47a

B. At $z = H$: $p_1 = p_2$ and $v_{z1} = v_{z2}$. 2.47b

C. At $z = H + h$: $v_{z2} = 0$. 2.47c

D. At large r, the fields tend to zero. 2.47d

E. At large r, the fields represent diverging radiation, if radiation exists. 2.47e

We shall now show that the method of orthogonal modes is not applicable in this problem. Following the established procedure, we assume that the pressure fields have the separated forms:

$$p_1 = R_1(r)F_1(z) \quad ; \quad p_2 = R_2(r)F_2(z).$$

The homogeneous forms of wave equations 2.45 are thereupon reduced to ordinary differential equations of the form:

$$\frac{d^2R_s}{dr^2} + \frac{1}{r}\frac{dR_s}{dr} + k_s^2 R_s = 0 \;,\quad s = 1, 2$$

$$\frac{d^2F_s}{dz^2} + \beta_s^2 F_s = 0 \quad ;$$

provided that:

$$\beta_s^2 = \frac{\omega^2}{c_s^2}(1 - iL_s) - k_s^2 \,. \tag{2.48}$$

The F-functions are first chosen to be solutions which fit the specifications at $z = 0$ and $H + h$:

$$F_1 = A \sin \beta_1 z$$

$$F_2 = B \cos \beta_2 (z - [H+h]) ;$$

the unknown constants will be evaluated from the boundary conditions at $z = H$. It is sufficient to assume that a constant factor lies in the R-functions, and take the constant A to be unity. The continuity conditions at H are satisfied when:

$$k_1 = k_2 \;,\; R_1 = R_2 \,,$$

$$B = \frac{\sin \beta_1 H}{\cos \beta_2 h} \,,$$

at values of β_1 and β_2 which satisfy the transcendental equation:

$$\frac{\tan \beta_1 H}{\beta_1} = \frac{\rho_2 (1 - iL_2)}{\rho_1 (1 - iL_1) \beta_2 \tan \beta_2 h} \,, \tag{2.49}$$

where, from equations 2.48,

$$\beta_2^2 = \beta_1^2 - \frac{\omega^2}{c_1^2}(1 - iL_1) + \frac{\omega^2}{c_2^2}(1 - iL_2) . \tag{2.50}$$

The orthogonality properties of the F-functions can be calculated directly:

$$\int_0^{H+h} F_m F_n \, dz = \left(1 - \frac{\rho_1 (1 - iL_1)}{\rho_2 (1 - iL_2)}\right)\left(\frac{\beta_{1n} \sin \beta_{1m} H \cos \beta_{1n} H - \beta_{1m} \cos \beta_{1m} H \sin \beta_{1n} H}{\beta_{1m}^2 + \beta_{1n}^2}\right) , \tag{2.51}$$

where the final result contains the substituted equation 2.49. It is clear that the functions are not orthogonal unless the complex densities are identical. For the lossless system, the modes are orthogonal only in the absence of a density discontinuity. This situation arises from the peculiar behavior of the fields at a medium discontinuity as discussed in Chapter II. The method of orthogonal modes is therefore not generally applicable for the two-layered medium, and we turn our attention to the solution by the Fourier-Bessel transformation.[1]

The point source term in equation 2.45a is again resolved into a continuum of plane sources and the solution of the pressure wave equations due to the stimulation of the plane source of wave number k_1 is sought:

$$\nabla^2 p_1 + \frac{\omega^2}{c_1^2}(1-iL_1)p_1 = -i\omega\rho_1(1-iL_1)\,\delta(z-d)\,k_1 J_0(kr) \qquad 2.52a$$

$$\nabla^2 p_2 + \frac{\omega^2}{c_2^2}(1-iL_2)p_2 = 0 \qquad 2.52b$$

The pressure fields are expressed as:

$$p_1 = p_{1P} + p_{K} \qquad 2.53a$$

$$p_2 = p_{2C} \qquad 2.53b$$

where p_{1P} and p_{1C} are the particular and complementary components of the total field in the upper medium. The field in the lower medium has no particular solution since there is no source present there.

The particular solution can be chosen to be the same as that derived previously:

$$p_{1P} = -i\omega\rho(1-iL_1)\frac{e^{\pm i\beta_1(z-d)}}{i\beta_1}k_1 J_0(k_1 r), \quad \begin{matrix} + : 0 \leq z \leq d \\ - : d \leq z \leq H \end{matrix} \qquad 2.29$$

provided
$$\beta_1^2 = \frac{\omega^2}{c_1^2}(1-iL_1) - k_1^2, \qquad 2.54$$

and the complementary solutions can be chosen as follows:

[1] Weighted orthogonality conditions of the type found in the familiar vibrating membrane problem are also excluded here, since the pressure field is not everywhere twice differentiable.

$$p_{1c} = Me^{i\beta_1 z} J_0(k_1 r) + Ne^{-i\beta_1 z} J_0(k_1 r) \qquad 2.55$$

$$p_{2c} = M' \cos\beta_2 (z - [H+h]), \qquad 2.56$$

provided $$\beta_2^2 = \frac{\omega^2}{c_2^2}(1 - iL_2) - k_2^2. \qquad 2.57$$

The field of the point source can then be found in the same manner as before. and expressed in a contracted form:

$$p_1 = \int_0^\infty \frac{2D_1}{\beta_1 \Psi} J_0(kr) k\,dk \sin\beta_1 z \left[\cos(\beta_1 d)\Psi + \beta_2 \sin\beta_1 d \sin\beta_1 (H+h)\right], \quad 0 \leq z \leq d$$

$$p_1 = \int_0^\infty \frac{2D_1}{\beta_1 \Psi} J_0(kr) k\,dK \sin\beta_1 d \left[\cos(\beta_1 z)\Psi + \beta_2 \sin\beta_1 z \sin\beta_1 (H+h)\right], \quad d \leq z \leq H \qquad 2.58$$

$$p_2 = \int_0^\infty \frac{2D_2}{\Psi} J_0(kr) K\,dK \sin\beta_1 d \cos\beta_2 [z - (H+h)], \quad H \leq z \leq H+h$$

$$\Psi = \frac{D_2}{D_1}\beta_1 \cos\beta_1 H \cos\beta_2 h - \beta_2 \sin\beta_1 H \sin\beta_2 h, \quad D_1 = -i\omega\rho_1(1 - iL_1), \quad D_2 = i\omega\rho_2(1 - iL_2).$$

If the medium were completely homogeneous, the above representations would reduce to the integrals found for the case of the single homogeneous layer, equations 2.35.

The remainder of this derivation will be concerned only with the completely lossless medium.

The above integrals can be carried into the complex γ-plane in the same manner as the case of the single homogeneous layer. The major difference is the four-valuedness of the integrands since there exist four possibilities of sign combinations for the β_1 and β_2 parameters. We shall choose the positive signs for the roots for the same reasons discussed before; thus:

$$\text{For } k < \frac{\omega}{c_1},\ \beta_1 = \left(\frac{\omega^2}{c_1^2} - k^2\right)^{1/2}; \quad k > \frac{\omega}{c_1},\ \beta_1 = -i\left(k^2 - \frac{\omega^2}{c_1^2}\right)^{1/2}; \qquad 2.59a$$

$$k < \frac{\omega}{c_2},\ \beta_2 = \left(\frac{\omega^2}{c_2^2} - k^2\right)^{1/2}; \quad k > \frac{\omega}{c_2},\ \beta_2 = -i\left(k^2 - \frac{\omega^2}{c_2^2}\right)^{1/2}. \qquad 2.59b$$

To insure that the integrands along the infinite arcs vanish for <u>all</u> ranges and depths, excluding the radial origin, the branch cuts are chosen so that the imaginary parts of both β_1 and β_2 are negative over the entire right half γ-plane. The cuts are shown in Figure 8.

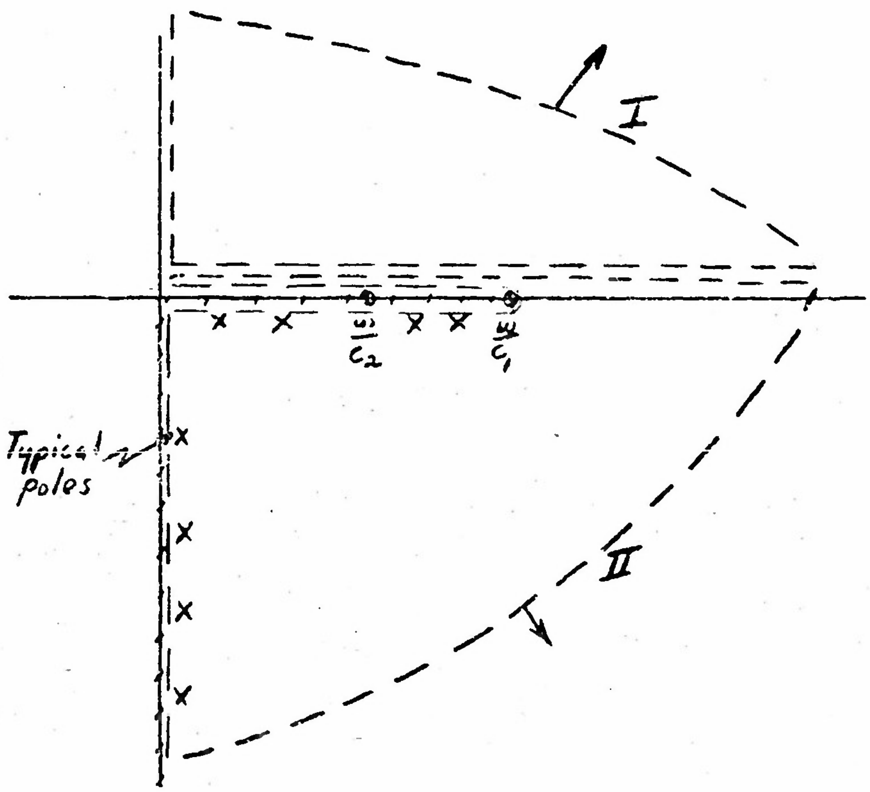

Figure 8.

The poles of the integrand are found at the zeros of ψ, which is precisely the determinantal equation for the distribution parameters of source-free modes, equation 2.49. In the lossless case, the transcendental equation $\psi=0$ is comparatively easy to solve. It is not necessary to look for complex β since it is proved generally in Part 3 that there can exist no complex β in a completely lossless medium in which the fields vanish at large z . The roots in the right half plane that do exist fall into three

types:

A. A finite number with γ real, $\omega/c_2 < k < \omega/c_1$, β_1 real, β_2 negative imaginary.

B. A finite number with γ real, $0 < k < \omega/c_2$, β_1 and β_2 real.

C. An infinite number with γ negative imaginary, β_1 and β_2 real.

The roots are demarcated in Figure 8 at typical locations.

The Bessel Function is transformed into the sum of the Hankel functions, and the contours I and II of Figure 8 are chosen. All of the line integrals vanish in the same manner as shown for the single layer: only these encircling the poles are non-zero. In particular, the integrals over the branch line on the real axis disappear because the integrands are even with respect to both β_1 and β_2. The final form of the solution is then the simple sum of the residues:

$$p_1 = 2\pi i D_1 \sum_{n=1}^{\infty} \frac{\beta_{2n}\gamma_n H_0^{(2)}(\gamma_n r)}{\beta_{1n} \left.\frac{\partial\psi}{\partial\gamma}\right|_{\gamma=\gamma_n}} \sin\beta_1 d \sin\beta_1 z \sin\beta_1(H+h), \quad 0 \leq z \leq H \tag{2.60}$$

$$p_2 = 2\pi i D_2 \sum_{n=1}^{\infty} \frac{\gamma_n H_0^{(2)}(\gamma_n r)}{\left.\frac{\partial\psi}{\partial\gamma}\right|_{\gamma=\gamma_n}} \sin\beta_1 d \cos\beta_2[z-(H+h)], \quad H \leq z \leq H+h$$

The field conditions at large ranges are, of course, satisfied by the presence of the Hankel Functions. The series converges for all depths and ranges, except for r=0.

The solution will now be discussed in detail.

1. The power of the Fourier-Bessel integral method of solution is clearly demonstrated in this problem, since the non-orthogonal character of the modes in the two-layered system obviated the use of the orthogonal mode technique. There does exist, however, a set of relations among the modes in a multi-layered medium which are analogous to the usual orthogonality relations applicable for the single-layered system. They are derived for the general

case in Chapter IV.

The integral transformation method again ensures that the modes form a complete set since only the line integrals on the residue contours are non-vanishing.

An examination of the solution reveals that the modes independently satisfy all the boundary conditions. The solution is also independent of the choice of branch cut.

2. The dependence of the modes upon the position co-ordinates closely resembles the form of the modes in the single layer problem. On the other hand, the distribution parameters β_{1n} and β_{2n} have a very complicated dependence upon the frequency and medium parameters, through the determinantal equation 2.49. This behavior makes impossible the identification of the modes as frequency independent entities in a fixed physical configuration.

It should be explicitly noted that the radial dependence of the modal field is identical in both layers. It is, therefore, proper that the k_n rather than the β_n be termed the "eigenvalues" of the wave equation.

One other major difference between the form of the modes in the one- and two-layered systems is the appearance of an additional stimulation factor, $\dfrac{1}{\left.\frac{\partial \psi}{\partial \gamma}\right|_{\gamma=\gamma_n}}$, which, since it is a function only of β_n, does not merit further attention. These preceeding comments are also applicable to the mode solution for the next problem.

3. The similarity of the modal forms in the one- and two-layered cases allows the physical interpretations of modes given on page 31 to be applied here, with some modification. Restricting the discussion to the completely lossless medium, we find that the freely propagating modes represent two independent forms. The first form, Type B, page 40, represents elementary

plane waves with $0 < k_n < \omega/c_2$ which are propagating at angles of incidence less than critical angle, defined as θ_c = arc sin c_1/c_2. These waves travel at the angle θ_{1n} = arc sin $k_n c_1/\omega$ in the upper layer, refract into the lower layer at the angle θ_{2n} = arc sin $k_n c_2/\omega$, and are completely reflected from the top and bottom surfaces. Typical ray paths are given in Figure 9. It is noted that the vertical field distribution is a purely sinusoidal function.

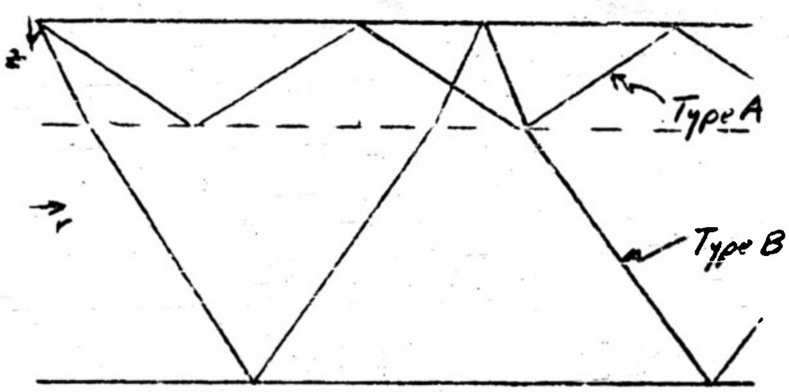

Figure 9.

The second form of undamped mode, Type A, page 40, represents elementary plane waves with $\omega/c_2 < k_n < \omega/c_1$ which are trapped in the upper layer as shown in Figure 9. The waves are propagating at angles greater than critical angle, undergoing total reflection at both the interface and the top surface. Trapping is further evidenced by the fact that the modal field dependence upon the depth co-ordinate is a purely hyperbolic function in the bottom layer, indicating destructive interference. For both modal types, active power is transferred only along the range co-ordinate and not across the horizontal cross-section.

Modes whose propagation factors lie on the imaginary axis, Type C, are purely attenuating radially and carry no active energy.

4. A Type B mode has cut-off frequency characteristics similar to a mode in the single homogeneous layer. Considered a function of decreasing frequency, the set of interfering plane waves travels at progressively smaller angles, ranging from critical angle to normal incidence. Below the cut-off frequency where $k_n = 0$, the propagating mode disappears and a new pole on the imaginary axis arises as the nth propagation factor. The mode is identified, as the frequency is changed, only by its sequence in the ordered series of poles of the field integrands, and not by its vertical field distribution. The Type A mode, considered a function of decreasing frequency, represents interfering plane waves which propagate in the upper layer at angles ranging from grazing incidence to critical angle. When the frequency is lowered beyond critical angle, where $k_n = \omega/c_2$, the mode disappears, with <u>no subsequent appearance of a new pole on the imaginary axis</u>. This mode simply drops out of existence beyond the "cut-off" frequency; cut-off frequency for the Type A mode is thus defined as the frequency for which $k_n = \omega/c_2$. The proof that the Type A mode cannot exist below cut-off in the general case is given in Chapter IV.

5. The phase velocity-frequency characteristic of the Type B modes is a monotonically decreasing function from $v_n = \infty$ at cut-off to c_2 at the high frequencies. The Type A modes have a similar behavior except that $v_n = c_2$ at cut-off, and approaches c_1 at the high frequencies.

6. The results can be extended qualitatively to the general case of a multi-layered medium which is bounded between perfectly reflecting planes. In all cases, the field of an arbitrary source can be found in terms of a complete set of modes, which are not orthogonal in the presence of a density discontinuity. Since the modes independently satisfy the source-free wave

equations and all the boundary conditions, one can find the modal dependence on the position co-ordinates by solving a transcendental equation of the type resembling equation 2.49. The amplitude factors must be found by recourse to the Fourier-Bessel integral transformation. In general, a finite number of modes represents waves propagating between the outer walls, a finite number represents trapped energy between layers of large velocity parameters, and an infinite number of strongly attenuating modal fields forms the remainder of the series.

<u>The Two-Layered Semi-Infinite Medium.</u>

The schematic configuration of the two-layered semi-infinite medium, which was the primary concern of Dr. Pekeris, is shown in Figure 10. The lower layer represents a liquid bottom which extends to $z=\infty$. The medium parameters in the lower medium are assumed to be greater than the corresponding parameters in the upper layer. A point source is present in upper layer at the location (0,d).

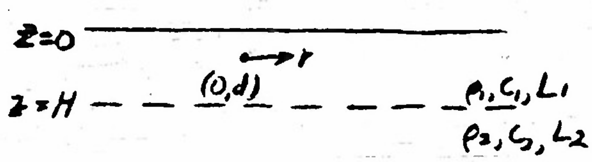

Figure 10.

The major physical difference between this problem and the preceding ones is that energy may leave the system through the bottom as well as along the radial co-ordinate.

The exact statement of the problem is the solution of the pressure wave equations in each of the layers:

$$\nabla^2 p_1 + \frac{\omega^2}{c_1^2}(1-iL_1)p_1 = -i\omega\rho_1(1-iL_1)\delta(r)\delta(z-d), \quad 0\leq z\leq H \tag{2.61a}$$

$$\nabla^2 p_2 + \frac{\omega^2}{c_2^2}(1-iL_2)p_2 = 0, \quad z\geq H \tag{2.61b}$$

and the corresponding velocity fields:

$$\underline{V}_1 = \frac{\nabla p_1}{-i\omega\rho_1(1-iL_1)} \qquad 2.62a$$

$$\underline{V}_2 = \frac{\nabla p_2}{-i\omega\rho_2(1-iL_2)}\,, \qquad 2.62b$$

subject to the boundary conditions:

A. At $z = 0$: $p_1 = 0$ 2.63a

B. At $z = H$, $p_1 = p_2$, $V_{z1} = V_{z2}$ 2.63b

C. At large z and r, the total field tends to 0. 2.63c

D. At large z and r, the fields represent diverging radiation, if radiation exists. 2.63d

It is first shown that the solutions of the source-free wave equation are generally non-orthogonal. We can easily find that the F-functions for this configuration are:

$$F_1 = \sin\beta_1 z \qquad , 0 \leqq z \leqq H \qquad 2.64a$$

$$F_2 = \sin\beta_1 H e^{i\beta_2 H} e^{-i\beta_2 z}, \quad z \geqq H\,, \qquad 2.64b$$

provided that $\beta_s^2 = \frac{\omega^2}{c_s^2}(1-iL_s) - k^2 \;, s=1,2\,,$ 2.65

and $\Psi \triangleq \frac{\beta_1 D_2}{D_1}\cos\beta_1 H + i\beta_2 \sin\beta_1 H = 0\,,\; D_s = -i\omega\rho_s(1-iL_s).$ 2.66

The F-functions are not orthogonal unless the complex densities are identical, since:

$$\int_0^{\infty} F_n F_m = i\left(\frac{\rho_1(1-iL_1)}{\rho_2(1-iL_2)} - 1\right)\frac{\sin\beta_{1n} H \sin\beta_{1m} H}{\beta_{2n}^2 + \beta_{2m}^2}\,. \qquad 2.67$$

The representation of the fields as Fourier-Bessel integrals can be found by the previously established method:

$$p_1 = \int_0^\infty \frac{D_1}{i\beta_1} e^{\pm i\beta_1(z-d)} J_0(kr)kdk + \int_0^\infty \frac{D_1}{i\beta_1} e^{-i\beta_1(z+d)} J_0(kr)kdk$$

$$+ \int_0^\infty \frac{i2D_1 e^{-i\beta_1 H}}{\beta_1 \Psi} \left(\frac{D_2}{D_1}\beta_1 - \beta_2\right) \sin\beta_1 d \sin\beta_1 z\, J_0(kr)kdk, \quad \begin{matrix} +: 0 \le z \le d \\ -: d \le z \le H \end{matrix} \qquad 2.68a$$

$$p_2 = \int_0^\infty \frac{2D_1}{\Psi} e^{i\beta_1 H - i\beta_2 z} \sin\beta_1 d\, J_0(kr)kdk, \qquad 2.68b$$

where Ψ is defined in equation 2.66.

When the medium is completely homogeneous, the last integral on right-hand side of equations 2.68a vanishes, and from the Sommerfeld transformation equation 2.34, the acoustic field reduces to the contributions from the source and its image in the surface:

$$p_1 \to p_2 \to -i\omega\rho(1-iL)\left(\frac{e^{-\left(\left[\frac{\omega^2}{c^2}(1-iL)\right]\left[r^2+(z-d)^2\right]\right)^{1/2}}}{\left[r^2+(z-d)^2\right]^{1/2}} - \frac{e^{-\left(\left[\frac{\omega^2}{c^2}(1-iL)\right]\left[r^2+(z+d)^2\right]\right)^{1/2}}}{\left[r^2+(z+d)^2\right]^{1/2}}\right) \qquad 2.69$$

Equation 2.68a can conveniently be expressed as the integrals:[1]

$$p_1 = \int_0^\infty \frac{2D_1}{\beta_1 \Psi} J_0(kr)kdk \sin\beta_1 z \left(\frac{D_2}{D_1}\beta_1 \cos\beta_1(H-d) + i\beta_2 \sin\beta_1(H-d)\right), \quad 0 \le z \le d$$

$$p_1 = \int_0^\infty \frac{2D_1}{\beta_1 \Psi} J_0(kr)kdk \sin\beta_1 d \left(\frac{D_2}{D_1}\beta_1 \cos\beta_1(H-z) + i\beta_2 \sin\beta_1(H-z)\right), \quad d \le z \le H \qquad 2.70$$

In the abbreviated form, the above equations read:

$$p = \int_0^\infty 2G_{1,2,3}(\beta_1,\beta_2) J_0(kr)kdk\,. \qquad 2.71$$

The remainder of the analysis deals only with the lossless configuration.

The Pekeris representation of the field integrals on the complex γ-plane

[1]The same ond result of Pekeris's, with the introduction of the complex parameters [12], p. 45.

will first be discussed.[1] As in the previous problem, there exist two branch points at $\beta_1 = \beta_2 = 0$. The cuts chosen by Pekeris are taken parallel to the imaginary axis on the Reimann sheet defined by the positive signs of the roots of the β parameters, where:

$$\text{For } K<\frac{\omega}{c_1},\ \beta_1=\left(\frac{\omega^2}{c_1^2}-K^2\right)^{1/2},\quad K>\frac{\omega}{c_1},\ \beta_1=-i\left(K^2-\frac{\omega^2}{c_1^2}\right)^{1/2};$$
$$K<\frac{\omega}{c_2},\ \beta_2=\left(\frac{\omega^2}{c_2^2}-K^2\right)^{1/2};\quad K>\frac{\omega}{c_2},\ \beta_2=-i\left(K^2-\frac{\omega^2}{c_2^2}\right)^{1/2}. \qquad 2.72$$

The γ-plane for this configuration is given in Figure 11.

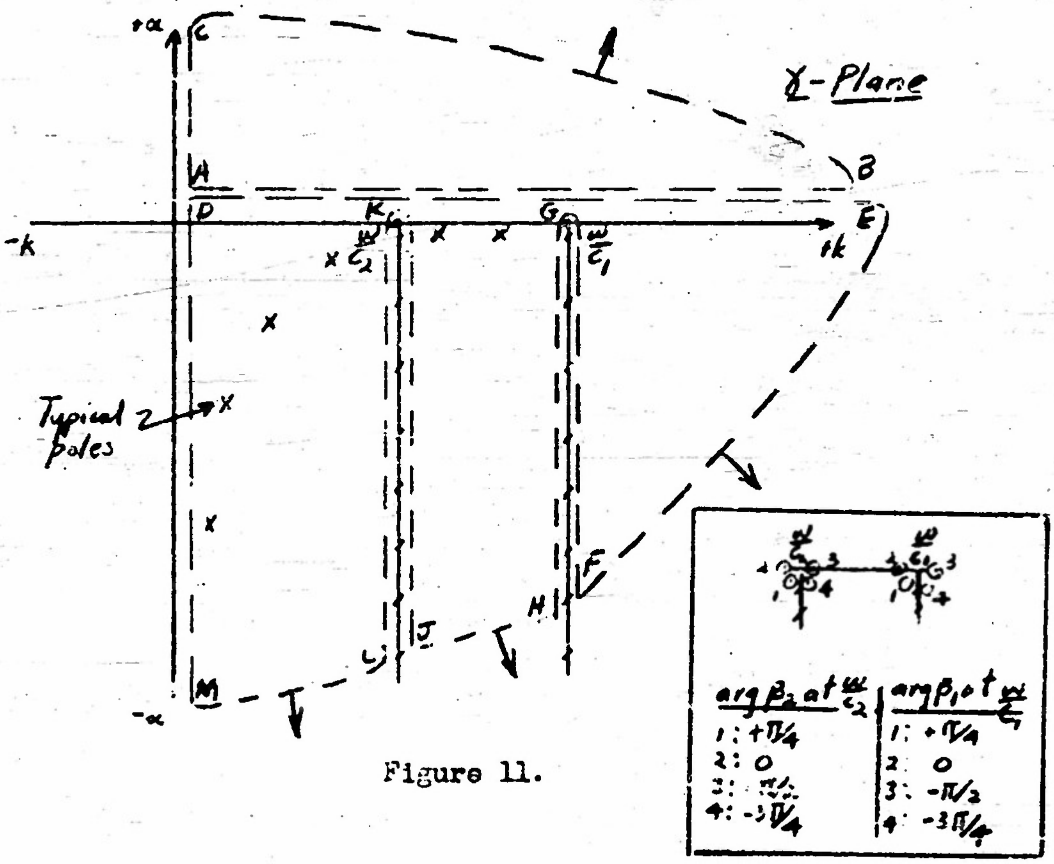

Figure 11.

The poles of the integrand are those values of γ for which $\mathcal{Y}=0$. These quantities are precisely those propagation factors of the modes found from the solution of the source-free wave equation, and the associated transcendental equation 2.66. The poles in the right-half γ-plane fall into two categories:

[1]Pekeris [12], Part 2.

A. A finite number on the real axis, $\gamma_n = k_n$, $\frac{\omega}{c_2} < k_n < \frac{\omega}{c_1}$; β_1 is purely real, β_2 negative imaginary.

B. An infinite number in the fourth quadrant located in the region defined by $k_n < \omega/c_2$. These poles have γ complex, β_1 and β_2 complex with positive imaginary parts. It is particularly noted that these poles exist even though the system is lossless. Dr. Pekeris did not introduce these poles in his formulation.

The poles are demarcated at typical locations in Figure 10.

Each of the integrals of equations 2.71 can be handled in the same manner. The Bessel function is transformed into the Hankel functions, referred to Figure 11, the paths are defined as follows:

$$\int_0^{\infty} G(\beta_1,\beta_2) H_0^{(1)}(kr) k\,dk = \int_{AC+CB} G(\beta_1,\beta_2) H_0^{(1)}(\gamma r)\gamma\,d\gamma \qquad 2.73a$$

$$\int_0^{\infty} G(\beta_1,\beta_2) H_0^{(2)}(kr) k\,dk = \int_{DM+ML+LK+KJ+JH+HG+GF+FE+\text{Poles}} G(\beta_1,\beta_2) H_0^{(2)}(\gamma r)\gamma\,d\gamma. \qquad 2.73b$$

The integrands along the paths CB and FE vanish as the radius of the circular segments extends to infinity. The integrand along the path MLJH will vanish only when the range is at some value greater than the depth; this fact arises upon examination of the original integrals in 2.68 where it is observed, from Figure 11, that the imaginary parts of both the β parameters are positive over this path.[1] The integrands along the paths vanish AC and DM as before. The integrals along HG and GF cancel because the integrands are single valued with respect to β_2 along these paths, and are even with respect to β_1 which possesses a simple sign change on opposite sides of the cut. The integrals along the branch cut through ω/c_2 do <u>not</u> cancel, however, because the integrands are mixed functions of β_2. The transformation of equations 2.71 then takes the

[1] To be precise, $\mathrm{Im}(\beta_1)$ and $\mathrm{Im}(\beta_2)$ are positive on ML, $\mathrm{Im}(\beta_2)$ is negative and $\mathrm{Im}(\beta_1)$ is positive on JH.

form:

$$p = 2\pi i \sum_n \mathrm{Res}_n \left[G(\beta_{1n}, \beta_{2n}) H_0^{(2)}(\gamma_n r)\gamma_n \right] + \int_{iK+KJ} G(\beta_1, \beta_2) H_0^{(2)}(\gamma r)\gamma d\gamma. \qquad 2.74$$

The difficulties with the Pekeris field representation are observed to be these:

1. The solution is only applicable for ranges which are somewhat greater than the depth. This situation is not only undesirable from a practical viewpoint, but it appears to be incompatible with the fact that the solution for the bounded systems of the previous sections was valid everywhere except at the origin.[1]

2. Type B modes contain β_{1n} and β_{2n} with positive imaginary parts. The residue of such a mode will contain the factor $e^{+\alpha z}$; the mode thus represents a field which is exponentially increasing with depth.[2] Such peculiar modal fields are definitely admissable in the Pekeris formulation, since the condition that the field vanish at infinite depth is applied only the total field representation. It should be explicitly stated that these modes do not contradict the energy conservation requirements stated in Chapter IV. Furthermore, the presence of a viscous loss, to the first order approximation stated in Chapter II, will not cause the fields of these modes to vanish at infinite depths.

3. Dr. Pekeris has shown that, under certain conditions, the branch line integral has the asymptotic form at large ranges of a modified dipole field. At first, it seems reasonable to ascribe this physical reality to the branch line integral, since such a field would be expected when the undamped modes are all "cut-off." However, if one considers the Type B modes as an integral

[1] Furthermore, there is the question of the validity of the Fourier-Bessel transformation of the radial delta function when the solution is not everywhere applicable.

[2] In the bottom.

part of the solution (which Pekeris does not), then it is <u>not</u> clear what a branch-line integral represents. The situation is this: The undamped (Type A) modes represent energy trapped in the upper layer. The Type B modes represent the acoustic energy which enters the bottom layer, and comprise an infinite set. All the physical expectations are thus accounted for. What added physical feature is presented by the non-vanishing branch-line integral?

These difficulties are all removed when the branch cuts are taken so that β_1 and β_2 have imaginary parts which are negative over the entire right half of the γ-plane. These cuts are along the lines $\mathrm{Imag}[\beta_1]=\mathrm{Imag}[\beta_2]=0$, and are identical with the cuts in Figure 8 for the bounded two-layered medium. The only poles that are present on this sheet of the Reimann surface are the Type A poles previously specified. The Type B poles certainly exist but they are now located on the other (non-physical) sheets of the Reimann surface, and consequently do not appear in this solution. The field integrals can now be evaluated in the same manner as given previously. Each of the integrals along the paths of Figure 8 vanishes, except those along the branch cut $0 < k < \omega/c_2$. The final result for s residues is:

$$p_1 = 2\pi i D_1 \sum_{n=1}^{s} \frac{k_n \beta_{2n}}{\beta_{1n} \left.\frac{\partial \psi}{\partial k}\right|_{k=k_n}} \sin\beta_{1n} d \sin\beta_{1n} z \, H_0^{(2)}(k_n r)$$
$$+ \int_0^{\omega/c_2} \frac{-2i D_2 \beta_2}{|\psi|^2} \sin\beta_1 d \sin\beta_1 z \, H_0^{(2)}(kr) k\,dk \quad , \; 0 \leqq z \leqq H \qquad 2.75a$$

$$p_2 = 2\pi i D_2 \sum_{n=1}^{s} \frac{k_n}{\left.\frac{\partial \psi}{\partial k}\right|_{k=k_n}} \sin\beta_{1n} d \, e^{-i\beta_{2n}(z-H)} H_0^{(2)}(k_n r)$$
$$+ \int_0^{\omega/c_2} \frac{-2i D_2 k}{|\psi|^2} \sin\beta_1 d \left(\beta_2 \sin\beta_1 H \cos\beta_2[H-z] - \beta_1 \frac{D_2}{D_1} \cos\beta_1 H \sin\beta_2[H-z]\right) H_0^{(2)}(kr)\,dk, \qquad 2.75b$$

$z \geqq H.$

The solution converges for all r and all z, where the convergence of the branch-line integral is justified on analytical grounds given later.

The solution is now discussed in detail.

1. We are now able to answer a basic point which Pekeris lest open for discussion, namely the reasons for the existence of the non-vanishing branch-line integral, and why it does not vanish in the two-layered semi-infinite medium which contains only a velocity discontinuity. First it is clear from the above derivation and the discussions of the preceding problems that there is no mathematical restriction present which requires the solution to have the form of a sum of discrete modes; hence, there is no apriori reason for expecting such a solution in the unbounded medium. Secondly, the critical point which led to the non-zero branch-line integral was the integrand dependence upon the distribution factor β. In the semi-infinite medium, it is recalled that the integrands were even with respect to β_1. but mixed with respect to β_2. Comparing this fact with the situation in the two-layered medium with perfectly reflecting boundaries, we can explain this dependence on the physical basis that the presence of a downward traveling wave alone in the bottom of the semi-infinite medium appears as an asymmetrical term in the integrand. For a multi-layered medium bounded by perfectly reflecting planes, there exist both upward and downward traveling waves in each layer which are represented by a symmetrical dependence of the integrand upon both the β's. The appearance of the branch-line integral is only a consequence of the fact that the medium is unbounded in the z direction, and is not associated with boundary conditions at the layer interface. Third, the necessary presence of the branch-line integral is suggested from the

analogous quantum mechanical problem of the one dimensional finite potential well. The solution of this problem has the form of a sum over a finite number of discrete modes (energy levels) plus an integration over a continuum of "modes." Fourth, we can ascribe a realistic physical interpretation to the branch line integral, to be discussed in a later paragraph. This characterization lends considerable justification for the validity of the field representation in the form given in equations 2.75.

2. The set of discrete modes is obviously not complete. As described later, these modes possess cut-off properties similar to the Type A modes of the previous section. There exists a frequency below which all of the poles on the real γ axis disappear, and no new poles appear on the imaginary axis. An arbitrary field distribution thus cannot be expressed solely by a sum of discrete modes; the branch line integral is present to account for the complete solution.

It is thus possible to have a solution for the Pekeris problem in which the discrete modes are orthogonal (by eliminating the density discontinuity) and are not complete; this fact lies in contradistinction to the solution for the case of the single homogeneous layer.

3. There exists an infinite number of possible representations of the solution, since the branch line choice is essentially arbitrary. We know, however, that the solution should be unique. The difficulty is resolved by the physical interpretation of the form of the solution.

It is recalled that, for the lossless case, the fields were originally represented by integrals over the real variable k. By extending the path of integration into the complex plane, we admitted the possibility of new field representations which are analytically correct but do not correspond

to physical reality, since these representations contain k as a complex variable. The clearest example of a non-physical component of the solution is the Type B mode, page 43, which is exponentially increasing in the bottom layer. The appearance of the non-physical forms of the solution is attributed to the inadequacy of the initial assumptions in Chapter II, particularly the neglect of shear waves in the fundamental equations. The solution as given in equations 2.75 is believed to be the only physically meaningful form, and in this sense is unique. Non-physical solutions for problems concerned with vertically bounded media are precluded since the final forms are essentially independent of the choice of branch cut.

It should not be inferred, however, that the Pekeris solution does not have value for calculation purposes. Dr. Pekeris, in fact, showed an adequate correspondence between some numerical integrations of the original field integrals, and the asymptotic behavior of his branch line integral.[1] Dr. Ide used the Type B modes to illustrate their theoretical correspondence to some experimental measurements in shallow water.[2] However, Ide treated these modes as though they were physically meaningful, an action which does not seem justified. His misinterpretation may have arisen from an error in the choice of signs in one of his introductory equations.[3] Ide's mode incorrectly represents a wave moving <u>upward</u> in the bottom layer with vanishing amplitude at large depths, instead of a wave moving downward with exponentially increasing amplitude.

4. The branch line integral of equation 2.75 is, in a sense, a blend of the Type B modes and the Pekeris branch line integral into a physically realistic

[1] Pekeris [12], Figure 23.
[2] Ide [2], App. E.
[3] Ide [2], App. A, Equation 6.

component of the total field, described as follows: We first recall that the new branch line integral is taken over the real γ axis, $0 < k < \omega/c_2$. In terms of the plane wave concept of a mode, the branch line integral represents the summation of the continuum of plane waves propagating in the upper layer at angles which are less than critical angle, and undergo multiple reflections between the top surface and the interface. Energy is refracted into the bottom at each reflection from the interface. The individual waves, represented by the branch line integrand at a particular value of k, do not vanish at large depths, but the integrated effect of these waves vanishes at large depths (and ranges) since:[1]

$$\lim_{z \to \infty} \int_0^{\text{Constant}} \cos kz \, f(k) \, dk = 0$$

where f(k) is any reasonable function of k.

A detailed analysis of the approximate forms of this branch line integral is beyond the scope of this study. Numerical integration may not be too arduous, at least for low frequencies, and small ranges and depths.

5. The trapped modes, Type A, page 43, have the same general characteristics as the Type A modes in the bounded two-layered system. These modes represent sets of elementary plane waves which are propagating in the upper layer at discreet angles with respect to the vertical. The angles fall in the range between between grazing and critical angles. These waves are, consequently, totally reflected from both the top surface and the interface. The mode thus represents energy trapped in the upper layer, and propagates, without loss, in the radial direction. The field of such a mode is observed to have an exponentially decreasing amplitude with depth[2] and, as can be shown from the discussion in Chapter IV, does not deliver active energy across the

[1] Adler [1], App. 4.

[2] In the bottom.

interface. "Cut-off" occurs at the frequency for which $k_n = \omega/c_2$. Below the cut-off frequency, the mode disappears since no new poles arise on the imaginary axis of the γ-plane.

The phase velocity of this type of mode is quite similar to the corresponding modes in the bounded two layer medium, and need not be discussed further.

6. When the parameters in the upper layer are greater than those in the bottom layer, the solution, when derived in the form of equations 2.75, consists solely of the branch line integral component, since there are no poles of the field integrands in this case.

CHAPTER IV

SOME GENERAL PROPERTIES OF THE ACOUSTIC MODES

Introduction

The information contained in this chapter supplements the discussion of the three examples of acoustic modal solutions described in the preceding chapter. The analysis is based on the work of Adler in studies of electromagnetic wave propagation.[1] The results are applicable in the case of the general multi-layered medium when fields have the form of equations 2.75 in which each component vanishes at $z=\infty$. The symbols p_n and $\underline{v}_n$ represent the pressure and particle velocity field of a typical mode. A branch line integral can also be represented by the same symbols since double integrations over z and k, to be subsequently performed, are independent operations.

The next two sections of this part are devoted to the derivation of the general orthogonality conditions that exist among the modes. The lossy system is included insofar as the results have a simple physical meaning. The last section provides a physical interpretation of the propagation factors for the discrete modes in the multi-layered systems from energy considerations.

Some Useful Identities

Since the modal sum is a linear combination of the solutions of the source-free wave equation, it is certain that each mode is a solution of the field equations:

$$\nabla p_n = -i\omega\rho(1-iL)\underline{v}_n \qquad 1.9a$$

$$\nabla\cdot\underline{v}_n = -\frac{i\omega}{\rho c^2}p_n \qquad 1.9b$$

---

[1] Adler [1].

The medium constants are, of course, those appropriate for any particular layer. The following identities can thereupon be derived directly from these equations:

$$\nabla \cdot p_m \underline{v}_n = \frac{-i\omega}{\rho c^2} p_m p_n + \underline{v}_m \cdot \underline{v}_n \left(-i\omega\rho[1-iL]\right) \qquad 3.1a$$

$$\nabla \cdot p_m \underline{v}_n^{+} = \frac{i\omega}{\rho c^2} p_m p_n^{+} + \underline{v}_m \cdot \underline{v}_n^{+} \left(-i\omega\rho[1-iL]\right) \qquad 3.1b$$

[1]

$$\nabla \cdot \left(p_m \underline{v}_n - p_n \underline{v}_m\right) = 0 \qquad 3.1c$$

$$\nabla \cdot \left(p_m \underline{v}_n^{+} + p_n^{+} \underline{v}_m\right) = -2L\omega\rho\left(\underline{v}_m \cdot \underline{v}_n^{+}\right) \qquad 3.1d$$

$$\nabla \cdot \left(p_m \underline{v}_n + p_n \underline{v}_m\right) = 2 p_m p_n \left(\frac{-i\omega}{\rho c^2}\right) + 2 \underline{v}_m \cdot \underline{v}_n \left(-i\omega\rho[1-iL]\right) \qquad 3.1e$$

$$\nabla \cdot \left(p_m \underline{v}_n^{+} + p_n \underline{v}_m^{+}\right) = \frac{i\omega}{\rho c^2}\left(p_m p_n^{+} + p_m^{+} p_n\right) + \left(\underline{v}_m^{+} \cdot \underline{v}_n + \underline{v}_m \cdot \underline{v}_n^{+}\right)\left(-i\omega\rho[1+iL]\right) \qquad 3.1f$$

$$\nabla \cdot \left(p_m^{+} \frac{\partial \underline{v}_m}{\partial \omega} + \frac{\partial p_m}{\partial \omega} \underline{v}_m^{+}\right) = \frac{-i}{\rho c^2}\left(p_m p_m^{+}\right) - i\rho\left(\underline{v}_m \cdot \underline{v}_m^{+}\right), \; L = 0 \qquad 3.1g$$

Power Orthogonality Conditions

The power orthogonality conditions can be conveniently interpreted as specifications on the power outflow due to modal interaction. The starting point is the transfer of the familiar Poynting vector of electromagnetic wave theory to the acoustic wave case.

The instantaneous Poynting vector $\underline{S}$ (r,z,t) is a real quantity which represents the instantaneous acoustic energy transferred/unit area/second at the point (r,z) and at the time t, with a direction indicated by the space orientation of $\underline{S}$. From its definition,

$$\underline{S} = \left[Re\left(p e^{i\omega t}\right)\right]\left[Re\left(\underline{v} e^{i\omega t}\right)\right]. \qquad 3.2$$

---

[1] The superscript $^{+}$ denotes the complex conjugate.

It is observed that the Poynting vector has the same space orientation as the particle velocity vector. Substituting the complete set of modes for the fields in the above equation, we have:

$$\underline{S} = \left[Re\left(\sum_n p_n e^{i\omega t}\right)\right]\left[Re\left(\sum_m \underline{v}_m e^{i\omega t}\right)\right] = \left[\sum_n Re\left(p_n e^{i\omega t}\right)\right]\left[\sum_m Re\left(\underline{v}_m e^{i\omega t}\right)\right] \quad 3.3$$

We are interested in the total instantaneous energy/second outflow from the cylindrical volume V shown in Figure 12. The height of the cylinder extends over the entire vertical depth of the medium. The cylinder is hollow with radii r_1 and r_2 at the inner and outer surfaces, which are noted A_1 and A_2. The top and bottom plane surfaces of the volume are noted A_3 and A_4.

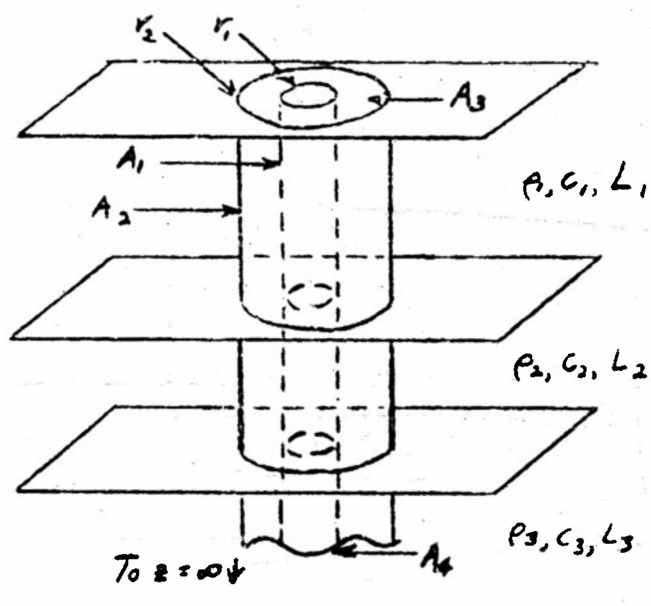

Figure 12.

It is assumed that no sources lie within V. The inner radius must be greater than zero because the acoustic field equations are singular at the radial origin. The total instantaneous power outflow from V is computed simply by integrating the vector $\underline{S}$ over the entire bounding surface:

$$\int \underline{S} \cdot d\underline{A} = \int_{A_1+A_2+A_3+A_4} \Big[\sum_n Re(p_n e^{i\omega t})\Big]\Big[\sum_m Re(\underline{v}_m e^{i\omega t})\Big] \cdot d\underline{A} \; . \qquad 3.4$$

There is no contribution from A_3 and A_4 since the Poynting vector is zero on these planes. The vertical component of particle velocity is tangential to surfaces A_1 and A_2. Equation 3.4 thus reduces to:

$$r_0 \int_{A_1+A_2} S_r \, dA = r_0 \int_{A_1+A_2} \Big[\sum_n Re(p_n e^{i\omega t})\Big]\Big[\sum_m Re(v_{rm} e^{i\omega t})\Big] dA \; . \qquad 3.5$$

The computation of the instantaneous real power transfer is facilitated when the modal fields are expressed as the following quantities:

$$Re(p_n e^{i\omega t}) = \frac{p_n}{2} e^{i\omega t} + \frac{p_n^{+}}{2} e^{-i\omega t} \qquad 3.6a$$

$$Re(v_{rm} e^{i\omega t}) = \frac{v_{rm}}{2} e^{i\omega t} + \frac{v_{rm}^{+}}{2} e^{-i\omega t} \qquad 3.6b$$

It is easily shown that the integrands of equation 3.5 have the form:

A. The "self-power" term:

$$Re\left(\frac{p_n v_{rm}^{+}}{2} + \frac{p_n v_{rm}}{2} e^{2i\omega t} \right) \qquad 3.7a$$

B. The "interaction-power" term:

$$Re\left(\frac{p_m v_{rn}^{+} + p_n^{+} v_{rm}}{2} + \frac{p_m v_{rn} + p_n v_{rm}}{2} \cdot e^{2i\omega t} \right) . \qquad 3.7b$$

Let us now consider the identity:

$$\nabla \cdot (p_m \underline{v}_n - p_n \underline{v}_m) = 0 \; . \qquad 3.1c$$

The divergence operation in cylindrical co-ordinates is explicitly:

$$\frac{\partial}{\partial z}\left(P_m V_{zn} - P_n V_{zm}\right) + \frac{1}{r}\frac{\partial r}{\partial r}\left(P_m V_{rn} - P_n V_{rm}\right) = 0 \tag{3.8}$$

where it is recalled that the velocity vector $\underline{V} = \underline{z}_0 V_z + \underline{r}_0 V_r$. The left hand term of equation 3.8 is everywhere defined and continuous; the second term is piecewise continuous when the tangential component of particle velocity is discontinuous at the interfaces. Equation 3.8 is integrated over the range of z, and the boundary conditions applied:

$$\left(P_m V_{zn} - P_n V_{zm}\right)\Big|_0^\infty = 0 = \int_0^\infty \frac{1}{r}\frac{\partial r}{\partial r}\left(P_m V_{rn} - P_n V_{rm}\right) dz \tag{3.9}$$

The left hand side vanishes for all r. It is necessary therefore that:

$$\int_0^\infty \left(P_m V_{rn} - P_n V_{rm}\right) dz = 0 \tag{3.10}$$

It is recalled that each mode is a solution of the <u>source-free</u> wave equation which satisfies all the boundary conditions. We have, therefore, the alternative possibility that the mode represents a converging wave. This situation is represented by a change of sign of the radial velocity component. The change in, for example, the mth mode produces the relation that:

$$\int_0^\infty \left(P_m V_{rn} + P_n V_{rm}\right) dz = 0 \tag{3.11}$$

The addition of 3.10 and 3.11 results in the final form of the time dependent power orthogonality condition, valid in the generally lossy case:

$$\int_0^\infty P_m V_{rn}\, dz = 0 \tag{3.12}$$

The time independent power orthogonality condition can be derived in the same manner, starting from identity 3.1d, but is valid only for the completely

lossy system:

$$\int_0^{\infty} p_m v_{rn}^{+} dz = 0 \quad , m \neq n. \qquad 3.13$$

In view of equation 3.12 and the form of the power flow terms of equations 3.7, it is clear that the time-varying parts of the interaction power outflow terms vanish. Thus the time-dependent part of the total real power outflow can be computed from the modal sum as though the modes were propagating independently. For a completely lossy system, equation 3.13 is valid: with the application of both equations 3.12 and 3.13, all the cross-product terms of equations 3.7b vanish, and the total time dependent power outflow is the sum of the "self-power" terms, equation 3.7a.

The time average real acoustic power outflow from the volume V can be found by averaging the total outflow over one period of the oscillation. Equations 3.7 thereupon reduce to:

A. "Self-power" terms: $Re\left(\frac{p_m v_{rm}^{+}}{2}\right)$ 3.14a

B. "Interaction-power" terms: $Re\left(\frac{p_m v_{rn}^{+} + p_n v_{rm}^{+}}{2}\right)$ 3.14b

In the lossy medium, the interaction terms give a definite contribution to the total real power outflow. On the other hand, these cross-product terms vanish in the lossless configuration since condition 3.13 is applicable, and the time average outflow can be computed from the sum of the time average "self-power" terms, equation 3.14a.

The single homogeneous medium bounded by perfectly reflecting walls is a special case. It is a simple matter to show, by direct integration of the wave equations, that both orthogonality conditions 3.12 and 3.13 are valid even if the medium is lossy. The cross-product terms in the power

outflow sum vanish in this situation, and outflow is computed from the simple sum of the "self-power" contributions.

Energy Orthogonality Conditions

We shall begin the derivation of the energy orthogonality conditions by deriving the "conservation of energy" theorems for the acoustic modes. Let us consider identity 3.1a:

$$\nabla \cdot (p_m \underline{v}_n) = \frac{-i\omega}{\rho c^2} p_m p_n - i\omega\rho(1-iL)\underline{v}_m \cdot \underline{v}_n \qquad 3.15$$

This equation is integrated over the volume in Figure 12, and the term on the left is transformed with the aid of Gauss's Law:

$$\int_{Area} p_m \underline{v}_n \cdot d\underline{A} + \int_V \omega\rho L \underline{v}_m \cdot \underline{v}_n \, dV = \int_V \frac{-i\omega}{\rho c^2} p_m p_n \, dV + \int_V -i\omega\rho \, \underline{v}_m \cdot \underline{v}_n \, dV \qquad 3.16$$

where ρ, c, L are now functions of z. The left hand side of the equation represents the instantaneous "complex" power leaving the volume through radiation and viscous loss. The right hand side represents the time rate at which the instantaneous potential and kinetic stored energies are decreasing within the volume. Identity 3.1b provides a similar relation for the time independent energies:

$$\int_{Area} p_m \underline{v}_n^+ \cdot d\underline{A} + \int_V \omega\rho L \underline{v}_m \cdot \underline{v}_n^+ \, dV = \int_V \frac{i\omega}{\rho c^2} p_m p_n^+ \, dV + \int_V -i\omega\rho \, \underline{v}_m \cdot \underline{v}_n^+ \, dV \qquad 3.17$$

It is observed that the time independent complex power outflow through radiation and dissipation is the difference between the time independent potential and kinetic energy storages.

The general time dependent energy orthogonality relations will first be derived. The left hand term of 3.16 disappears when n ≠ m upon application

of power orthogonality condition 3.12. We have then:

$$\iiint_V \left[\left(\frac{-i\omega}{\rho c^2} p_m p_n\right) - \left(i\omega\rho[1-iL]\right)\left(v_{zm} v_{zn} + v_{rm} v_{rn}\right)\right] dV = 0 \qquad 3.18$$

An alternative relation exists when the mth mode represents a converging wave where the sign of v_{rm} is changed:

$$\int_V \left[\left(\frac{-i\omega}{\rho c^2} p_m p_n\right) - \left(+i\omega\rho[1-iL]\right)\left(v_{zm} v_{zn} - v_{rm} v_{rn}\right)\right] dV = 0 \qquad 3.19$$

Adding and subtracting 3.13 and 3.19 produces the time dependent energy orthogonality conditions:

$$\int_V \left[\frac{p_m p_n}{\rho c^2} + \rho(1-iL) v_{zm} v_{zn}\right] dV = 0 \qquad 3.20a$$

$$\int_V \rho(1-iL) v_{rm} v_{rn}\, dV = 0 \qquad 3.20b$$

If we assume that the radii of the volume in Figure 12 are separated only by an infinitesimal distance, Taylor's theorem can be applied, and the above equations enter their final form as energy storages/unit radial length

$$\int_0^\infty \left[\frac{p_m p_n}{\rho c^2} + \rho(1-iL) v_{zm} v_{zn}\right] dz = 0 \qquad , m \neq n \qquad 3.21a$$

$$\int_0^\infty \rho(1-iL) v_{rm} v_{rn}\, dz = 0 \qquad , m \neq n \qquad 3.21b$$

The time independent energy orthogonality conditions can be developed in the same manner when the system is completely lossless. The results are these:

$$\int_0^\infty \left(\frac{p_m p_n^+}{\rho c^2} + \rho v_{zm} v_{zn}^+\right) dz = 0 \qquad 3.22a$$

$$\int_0^\infty \rho v_{rm} v_{rn}^+\, dz = 0 \qquad 3.22b$$

The bounded single homogeneous medium is the case in which the familiar orthogonality relations hold. For this configuration, it can be shown from

an integration of the wave equations that <u>each</u> of the terms of equations 3.21 and 3.22 is independently zero even if the medium is lossy.

We can now understand why the modes in the two-layered configurations are not orthogonal in the usual sense. From 3.21 and 3.22 it is clear that, although the <u>total</u> contribution to the energy storage from interacting modes is always zero (in the lossless case), it is not necessary that the potential and kinetic interaction energy contributions be <u>independently</u> zero. We can expect, however, on the basis of the discussion in Chapter III, that the usual orthogonality relations will hold in a multi-layered configuration when the medium has a uniform "complex" density.

<u>Physical Interpretation of the Propagation Factor from Energy Considerations</u>

The concern here is only with the discreet modes that may exist as components of the field representation in a multi-layered medium.

Let us consider the identity 3.1b which is written for the case m=n:

$$\nabla \cdot p_n \underline{v}_n^{+} = \frac{i\omega}{\rho c^2} p_n p_n^{+} - i\omega\rho(1 - iL)\underline{v}_n \cdot \underline{v}_n^{+} \qquad 3.23$$

This equation is integrated over the volume V of Figure 12, and the left hand term transformed by Gauss's Law:

$$\int_{A_1+A_2} p_n v_{rn}^{+} dA = \int_V \left(\frac{i\omega}{\rho c^2} p_n p_n^{+} - i\omega\rho\, \underline{v}_n \cdot \underline{v}_n^{+} - \omega\rho L\, \underline{v}_n \cdot \underline{v}_n^{+} \right) dV \qquad 3.24$$

where the integrals over the surfaces A_3 and A_4 vanish in view of the field values thereon. ρ, c, L are again functions of z. Equation 3.24 can also be expressed in the form:

$$Re \int_{A_1+A_2} p_n v_{rn}^{+} dA = -\int_V \omega\rho L\, \underline{v}_n \cdot \underline{v}_n^{+} dV \qquad 3.25a$$

$$Im \int_{A_1+A_2} p_n v_{rn}^{+} dA = \int_V \left(\frac{\omega}{\rho c^2} p_n p_n^{+} - \rho\, \underline{v}_n \cdot \underline{v}_n^{+} \right) dV \qquad 3.25b$$

Assume that the nth mode is diverging. At large ranges where the asymptotic form of the Hankel function is valid, the fields have the following forms:

$$p_n \simeq \frac{e^{-\alpha_n r - ik_n r}}{(k_n r)^{1/2}}(-i\omega\rho)\sin\beta_n z \quad , \quad \text{for small } \alpha_n \qquad 3.26$$

$$v_{rn} \simeq \frac{ik_n p_n}{-i\omega\rho} \qquad 3.27$$

The left hand term of 3.25a can thus be written as:

$$\int_{A_1+A_2} p_n v_{rn}^{+}\, dA = \int_{A_1+A_2} \phi(z)\frac{e^{-2\alpha_n r}}{r}\, dA. \qquad 3.28$$

where : $\phi(z)$ is a positive real quantity.

Over A_1 the particle velocity vector is anti-parallel with respect to the normal to the surface. Over A_2 this vector is parallel to the outward normal. The net power outflow is then:

$$\int_{A_1+A_2} p_n v_{rn}^{+}\, dA = -e^{-2\alpha_n r_1}\, 2\pi\int_0^\infty \phi(z)\,dz + e^{-2\alpha_n r_2}\, 2\pi\int_0^\infty \phi(z)\,dz \qquad 3.29$$

The second term on the right is always positive, proving that a diverging cylindrical wave does actually correspond to outgoing radiation. The first integral on the right hand side of the above equation is always negative. To satisfy equation 3.25a, which states that the energy in at A_1 must be greater than the energy out at A_2, α_n must have a negative sign. A similar analysis of the converging wave provides the general result that equation 3.25a is satisfied only when k_n and α_n have opposite signs. Thus no first and third quadrant poles will ever arise. Since the derivation of this theorem stemmed directly from the fundamental field relations, it is clear that the exclusion of waves that increase in the direction of propagation is not the result of an extra condition imposed on the solution.

The attenuation factor can be given an interpretation by further manipulation of equations 3.25. With reference to Figure 12, if r_2 is separated from r_1 by only an infinitesimal distance, Taylor's theorem provides this approximate form of 3.25a, applicable at large ranges:

$$2\alpha_n = \frac{\int_0^\infty \omega\rho L v_{rn} v_{rn}^{+} dz}{Re\int_0^\infty p_n v_{rn} dz} \qquad 3.30$$

Thus the attenuation factor represents the constant ratio of the average viscous power loss/unit radial length to the total power transferred through the radial cross section at a particular range.

An interesting incidental point is the ratio of real to reactive power in a time average "self-power" term for the completely homogeneous medium. From previous considerations, this term has the form:

$$p_n v_n^{+} = \frac{|p_n|^2\left[(\alpha_n L + k_n) + i(\alpha_n - k_n L)\right]}{\omega\rho} \qquad 3.31$$

For $k_n >> \alpha_n$ the ratio is approximately 1/L; for $k_n << \alpha_n$ the ratio is approximately L. When $k_n = \alpha_n$, the ratio is slightly greater than 1.

The remainder of this section is concerned only with the purely lossless medium.

We shall first prove the following theorem:

In a lossless multi-layered medium, a discreet solution of the source-free wave equation which vanishes at large z cannot have a complex propagation factor; if, in particular, the medium is unbounded in the z direction, the propagation factor must be purely real.

Let us assume that the qth mode has a complex propagation factor γ_q. The mode can be written in the form $p_q = F_q(z)R_q(r)$ where:

$$\frac{d^2F_q}{dz^2} + \beta_q^2(z)F_q = 0 \qquad 3.32$$

and, as usual:

$$\beta_q^2(z) = \frac{\omega^2}{c^2(z)} - \gamma_q^2 \qquad 3.33$$

The velocity parameter is now considered to be a function of the depth, but is, of course, constant over a given layer. The complex conjugates of the above equations are these:

$$\frac{d^2F_q^+}{dz^2} + \beta_q^{2+}F_q^+ = 0 \qquad 3.34$$

$$\beta_q^{2+} = \frac{\omega^2}{c^2} - \gamma_q^{2+} \qquad 3.35$$

We next multiply 3.32 and 3.34 by F_q^+ and F_q respectively, and integrate the two equations over the entire range of z. The integrands are not defined at the values of z where there is a discontinuity in the medium parameters, but the integrals nevertheless exist. The first term of both the equations can be integrated by parts:

$$\int_{-\infty}^{+\infty} F_q^+ \frac{d^2F_q}{dz^2}dz = F_q^+\frac{dF_q}{dz}\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty}\frac{dF_q^+}{dz}\frac{dF_q}{dz}dz \qquad 3.36$$

$$\int_{-\infty}^{+\infty} F_q \frac{d^2F_q^+}{dz^2}dz = F_q\frac{dF_q^+}{dz}\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty}\frac{dF_q^+}{dz}\frac{dF_q}{dz}dz \qquad 3.37$$

Since the field is zero at $|z| = \infty$, we find, after subtraction of the above equations, that:

$$\int_{-\infty}^{+\infty}(\beta_q^{2+} - \beta_q^2)F_q^+F_q\,dz = 0 \qquad 3.38$$

and

$$\beta^+ = \pm\beta \qquad 3.39$$

Since the distribution parameters β_q cannot be complex, the propagation factor γ_q cannot be complex, and the first part of the theorem has been proved.

We next consider the net modal power outflow from a cylindrical volume of the type given in Figure 12. The volume extends from z=oo to interface between the bottom two layers. A single mode is assumed to have a purely imaginary propagation factor, in which case the field is oscillatory with time, and has an exponential decay with range. The real power transfer across the cylindrical surfaces is zero as shown from equation 3.31. The fields, of course, are zero at infinite depth. There is a non-zero real power contribution from energy entering the top plane surface of the cylinder, since the mode has an $e^{-i\beta_n z}$ dependence in the bottom layer. The result is that a finite amount of energy flows into the cylinder, but no energy leaves or is dissipated as a viscous loss. This conclusion contradicts the conservation of energy equation 3.1b. The proof of the theorem is now complete.

For a discrete mode which exists in the lossless system, equation 3.30 reads:

$$\alpha_n \int_0^\infty p_n v_{rn}^+ dz = 0 \qquad 3.40$$

Thus either α_n or the time average integrated power transferred radially must vanish. If α_n is zero, then, from the above theorem, $k_n \neq 0$, and the mode represents an undamped cylindrical wave. A study of the acoustic Poynting vector for this case shows that such a mode carries only real time average power in the radial direction. When $\alpha_n \neq 0$, it is clear that only reactive power is carried by the field. The interpretation of α_n for this situation is found from equation 3.25b; it is ratio of the net time average energy stored/unit radial length to the total reactive power transferred/radian at a given range:

$$\alpha_n = \frac{\int_0^\infty \left(\frac{p_n p_n^+}{2\rho c^2} - \frac{\rho v_{rn} v_{rn}^+}{2} \right) dz}{Im \int_0^\infty \frac{p_m v_m^+}{\omega} dz} \qquad 3.41$$

We now turn our attention to the real propagation factor k_n. The sign of k_n, in conjunction with the time factor $e^{i\omega t}$, indicates the direction of an undamped cylindrical wave. By direct computation it was shown that the wave direction was also the direction of radiated energy. However, although ω/k_n is the velocity of the propagating wave, the velocity of energy travel will now be shown to be the group velocity, $\partial\omega/\partial k_n$. From identity 3.1g:

$$\nabla \cdot \left(p_m^+ \frac{\partial \underline{v}_m}{\partial \omega} + \frac{\partial p_m}{\partial \omega} \underline{v}_m^+ \right) = -i \left(\frac{p_m p_m^+}{\rho c^2} + \rho \underline{v}_m \cdot \underline{v}_m^+ \right) \qquad 3.1g$$

Let us integrate this equation over the volume V in Figure 12, applying the usual boundary conditions over A_3 and A_4:

$$\int_{A_1 + A_2} \left(p_m^+ \frac{\partial v_{rm}}{\partial \omega} + \frac{\partial p_m}{\partial \omega} v_{rm}^+ \right) dA = -i \int_V \left(\frac{p_m p_m^+}{\rho c^2} + \rho \underline{v}_m \cdot \underline{v}_m^+ \right) dV \qquad 3.42$$

We can proceed with the same method introduced earlier. The field quantities on the left hand side of the above equation are approximated by the use of the Hankel Function asymptotic form. If this term is now computed at ranges r_1 and r_2 which are separated by an infinitesimal distance then there results:

$$\frac{\partial \omega}{\partial k_n} = \frac{Re \int_0^\infty p_m v_{rm}^+ dz}{\int_0^\infty \left(\frac{p_m p_m^+}{2\rho c^2} + \frac{\rho \underline{v}_m \cdot \underline{v}_m^+}{2} \right) dz} \qquad 3.43$$

The group velocity is thus interpreted as the velocity of energy propagation in the nth mode since it represents the ratio of the total power outflow across a cylindrical cross section at given range to the energy stored/unit radial length.

REFERENCES

1. R. B. Adler, *Properties of Guided Waves on Inhomogeneous Cylindrical Structures*, Rad. Lab. Elect. Report No. 102, M.I.T. (1949).
2. J. M. Ide, R. F. Post, and W. L. Fry, *The Propagation of Sound at Low Frequencies*, NRL Report No. S-2113 (1943).
3. E. R. Lapwood, "The Disturbance due to a Line Source in a Semi-Infinite Elastic Medium," *Phil. Trans. Roy. Soc. Ser. A*, *242*, 63 (1949).
4. H. N. Marsh, Jr., *Theory of the Anomalous Propagation of Acoustic Waves in the Ocean*, USUSL Report No. 111 (1950).
5. M.I.T. R.L. Series, Volume 13, *Propagation of Short Radio Waves*, McGraw-Hill, N. Y. 1951.
6. P. M. Morse, *Vibration and Sound*, McGraw-Hill, N. Y. (1948).
7. P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*, Tech. Press, Cambridge (1946).
8. National Research Council, *Physics of Sound in the Sea*. Sum. Tech. Rep., Div. 6, *8* (1946).
9. National Research Council, *Principles of Underwater Sound*. Sum. Tech. Rep., Div. 6, *7* 1946).
10. M. Newlands, "The Disturbance due to a Line Source in a Semi-Infinite Elastic Medium with a Single Surface Layer," *Phil. Trans. Roy. Soc.*, *Ser. A*, *245*, 213 (1952).
11. L. Page, *Introduction to Theoretical Physics*, Van Nostrand, N.Y. (1935).
12. C. L. Pekeris, "Theory of Propagation of Explosive Sound in Shallow Waves," *Geo. Soc. of Amer.*, Mem. 27 (1948).
13. C. L. Pekeris, "Ray Theory vs. Normal Mode Theory in Wave Propagation Problems," *Proc. Sym. in App. Math.*, *2*, 71 (1950).

14. A. Sommerfeld, Partial Differential Equations in Physics, Academic Press, N. Y. (1949).

15. J. L. Stone, Acoustic Wave Propagation Through Inhomogeneous Media, Dept. of Elect. Eng'g., Princeton University, ONR Contract N6onr-27005, Tech. Rep. No. 7 (1951).

16. J. A. Stratton, Electromagnetic Theory, McGraw-Hill, N. Y. (1941).

17. J. L. Worzel and M. Ewing, "Explosive Sounds in Shallow Water," Geo. Soc. of Amer., Mem. 27, (1948).

DISTRIBUTION LIST FOR TECHNICAL REPORTS
Princeton University NR 384-201
Contract N6onr-27005

Research and Development Board
Pentagon Building
Washington 25, D. C. (1)

Chief of Naval Research
Office of Naval Research
Navy Department
Washington 25, D. C.
Attn: Acoustics Branch (2)

Director. Naval Research Laboratory
Washington 25, D. C.
Attn: Technical Information Officer (9)

Director
Office of Naval Research Branch Office
150 Causeway Street
Boston, Massachusetts (1)

Director
Office of Naval Research Branch Office
346 Broadway
New York, New York (1)

Director
Office of Naval Research Branch Office
1030 East Green Street
Pasadena, California (1)

Director
Office of Naval Research Branch Office
86 East Randolph Street
10th Floor John Crerar Library Bldg.
Chicago 1, Illinois (1)

Officer-in-Charge
Office of Naval Research
Navy #100, Fleet Post Office
New York, New York (1)

Director, U.S. Navy Underwater Sound
Reference Laboratory
Office of Naval Research
P. O. Box 3629
Orlando, Florida (1)

Director
Naval Research Laboratory
Washington 25, D. C.
Attn: Sound Division (1)

Director
U. S. Navy Electronics Laboratory
San Diego 52, California (1)

U. S. Naval Academy
Naval Postgraduate School
Physics Department
Monterey, California
Attn: Professor L. E. Kinsler (1)

Director
Naval Ordnance Laboratory
White Oak, Maryland
Attn: Sound Division (1)

Director
Marine Physical Laboratory at the
U. S. Navy Electronics Laboratory
San Diego 52, California (1)

Director
David Taylor Model Basin
Carderock, Maryland (1)

Director
Ordnance Research Laboratory
Pennsylvania State College
State College, Pennsylvania (1)

Chief, Bureau of Ships
Navy Department
Washington 25, D. C.
Attn: 847 (1)

Undersea Warfare Committee
Navy Department
Washington 25, D. C. (1)

Massachusetts Institute of Technology
Acoustics Laboratory
Cambridge 39, Massachusetts
Attn: Professor R.H. Bolt (1)

Harvard University, Cruft Laboratory
Department of Engineering Science
and Applied Physics
Cambridge, Massachusetts
Attn: Professor F.V. Hunt (1)

Distribution List (cont'd)

Catholic University of America
Washington 17, D. C.
Attn: Professor K. F. Herzfeld (1)

Scripps Institute of Oceanography
La Jolla, California
Attn: Dr. Roger Revelle (1)

Brown University
Department of Applied Physics
Providence 12, Rhode Island
Attn: Professor R. B. Lindsay (1)

University of California
Department of Physics
Los Angeles, California
Attn: Professor Rudnick (1)

Woods Hole Oceanographic Institute
Woods Hole, Massachusetts
Attn: Mr. Vine (1)

Commanding Officer
Naval Air Development Center
Johnsville, Pennsylvania (1)

University of Michigan
Ann Arbor, Michigan
Attn: Dr. R. M. Howe (1)

Lamont Geological Observatory
Torre Cliffs
Palisades, New York
Attn: Dr. J. Worzel (1)

Hudson Laboratory
Columbia University
145 Palisades
Dobbs Ferry, New York, N. Y.
Attn: Dr. E. T. Booth (1)

Department of the Air Force
Commanding Officer
Air Force Cambridge Research Center
230 Albany Street
Cambridge, Massachusetts
Attn: Dr. Norman Haskell (1)

Director
Defense Research Laboratory
University of Texas
Austin, Texas (1)